



# Angular Statistics of Lagrangian Trajectories in Turbulence

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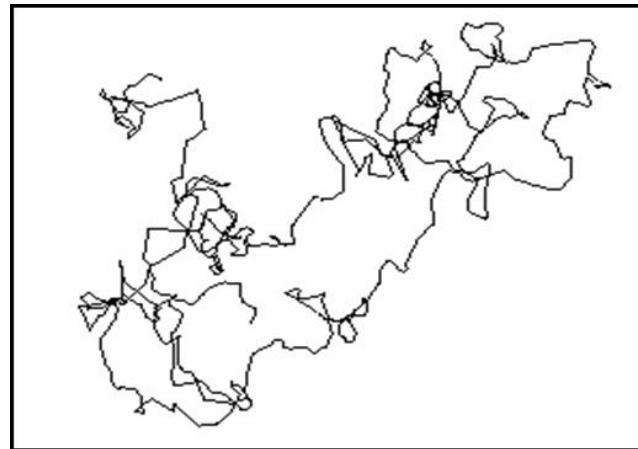
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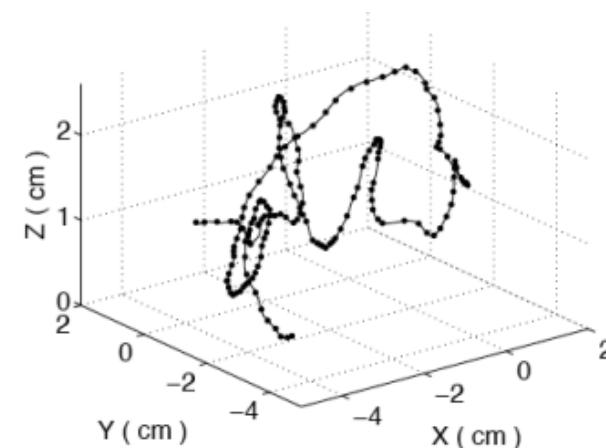
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## Turbulence in the Lagrangian frame

How do we recognize turbulence from a fluid-particle trajectory ?

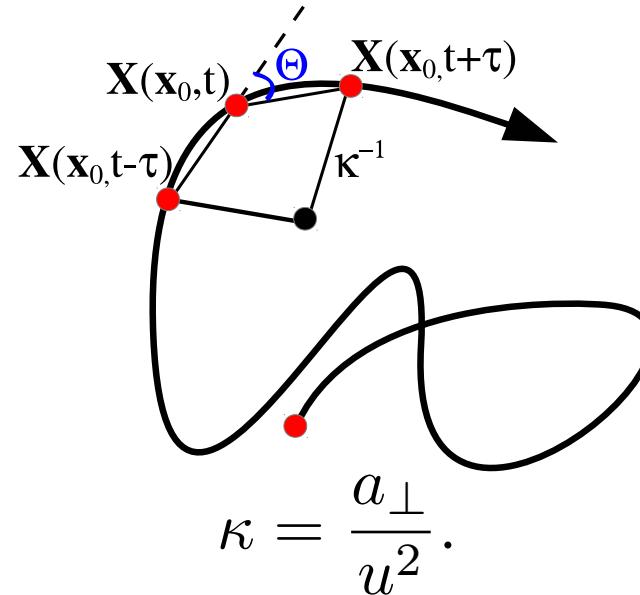


Random walk



Mordant et al. 2004

## Curvature measurements



Braun, De Lillo & Eckhardt (JoT 2006)

Xu, Ouellette & Bodenschatz (PRL 2007)

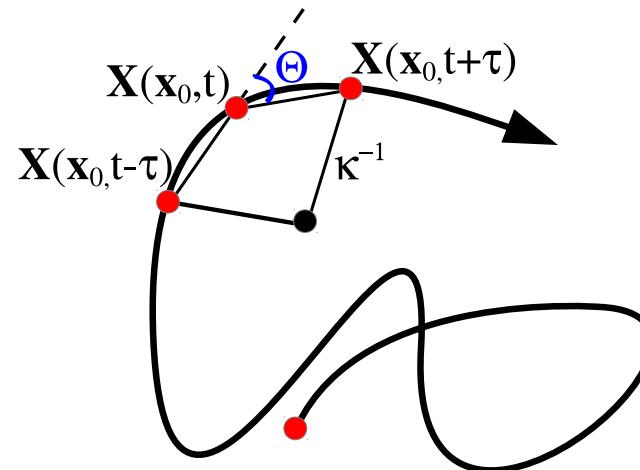
Focusing in particular on relation curvature and vorticity.

Reveal the multi-scale dynamics of turbulence from the curvature of the trajectories ?

Curvature is instantaneous. Consider coarse-grained curvature ?

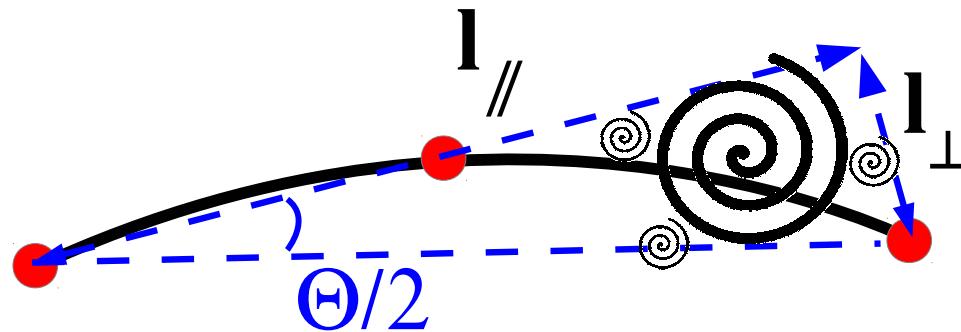
## Multi-timescale analysis

A new, related measure, the directional change (Burov et al. PNAS 2013).



$$\cos(\Theta(t, \tau)) = \frac{\delta \mathbf{X}(x_0, t, \tau) \cdot \delta \mathbf{X}(x_0, t + \tau, \tau)}{|\delta \mathbf{X}(x_0, t, \tau)| |\delta \mathbf{X}(x_0, t + \tau, \tau)|}. \quad (1)$$

## Behaviour of the mean-coarse-grained-angle, Short times



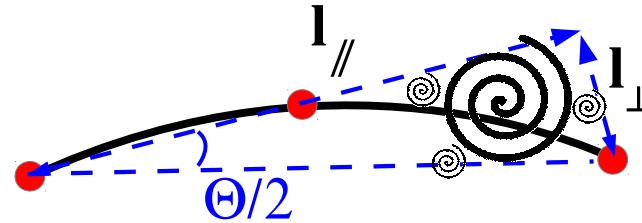
$$\Theta(t, \tau)/2 \approx l_{\perp}/l_{\parallel}$$

Taylor expansion :

$$\mathbf{X}(t + \tau) \approx \mathbf{X}(t) + \tau \dot{\mathbf{X}}(t) + (\tau^2/2) \ddot{\mathbf{X}}(t) + \mathcal{O}(\tau^3) \quad (2)$$

And,  $\dot{\mathbf{X}}(t) = \mathbf{u}(t)$ ,  $\ddot{\mathbf{X}}(t) = \mathbf{a}(t)$ .

## Behaviour of the mean-coarse-grained-angle, Short times



$$l_{\parallel} \approx 2\tau \|\langle \mathbf{u} \rangle_{\tau} \| \text{ and } l_{\perp} \approx 2\tau^2 \|\langle \mathbf{a}_{\perp} \rangle_{\tau} \|$$

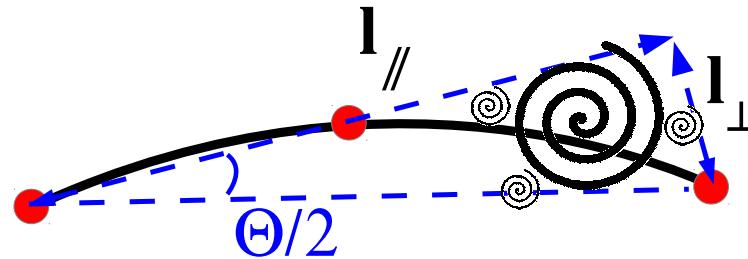
$$\langle |\Theta(t, \tau)| \rangle \approx 2\tau \left\langle \frac{\|\langle \mathbf{a}_{\perp} \rangle_{\tau} \|}{\|\langle \mathbf{u} \rangle_{\tau} \|} \right\rangle \quad (3)$$

so that, for  $\tau \ll \tau_K$ , assuming independence of velocity and acceleration,

$$\theta(\tau) \approx 2\tau \frac{\langle \|\mathbf{a}_{\perp}\| \rangle}{\langle \|\mathbf{u}\| \rangle} \sim \frac{\tau}{T} R_{\lambda}^{1/2} \quad (4)$$

Linear scaling for smooth (sub-Kolmogorov) flow

## Inertial range scaling



The acceleration induced by inertial range structures dominated by pressure gradient.

$$(\nabla p(l))^2 \sim \epsilon^{4/3} l^{-2/3} \rightarrow \quad (5)$$

Kolmogorov phenomenology :  $\tau \sim \epsilon^{-1/3} l^{2/3}$  therefore

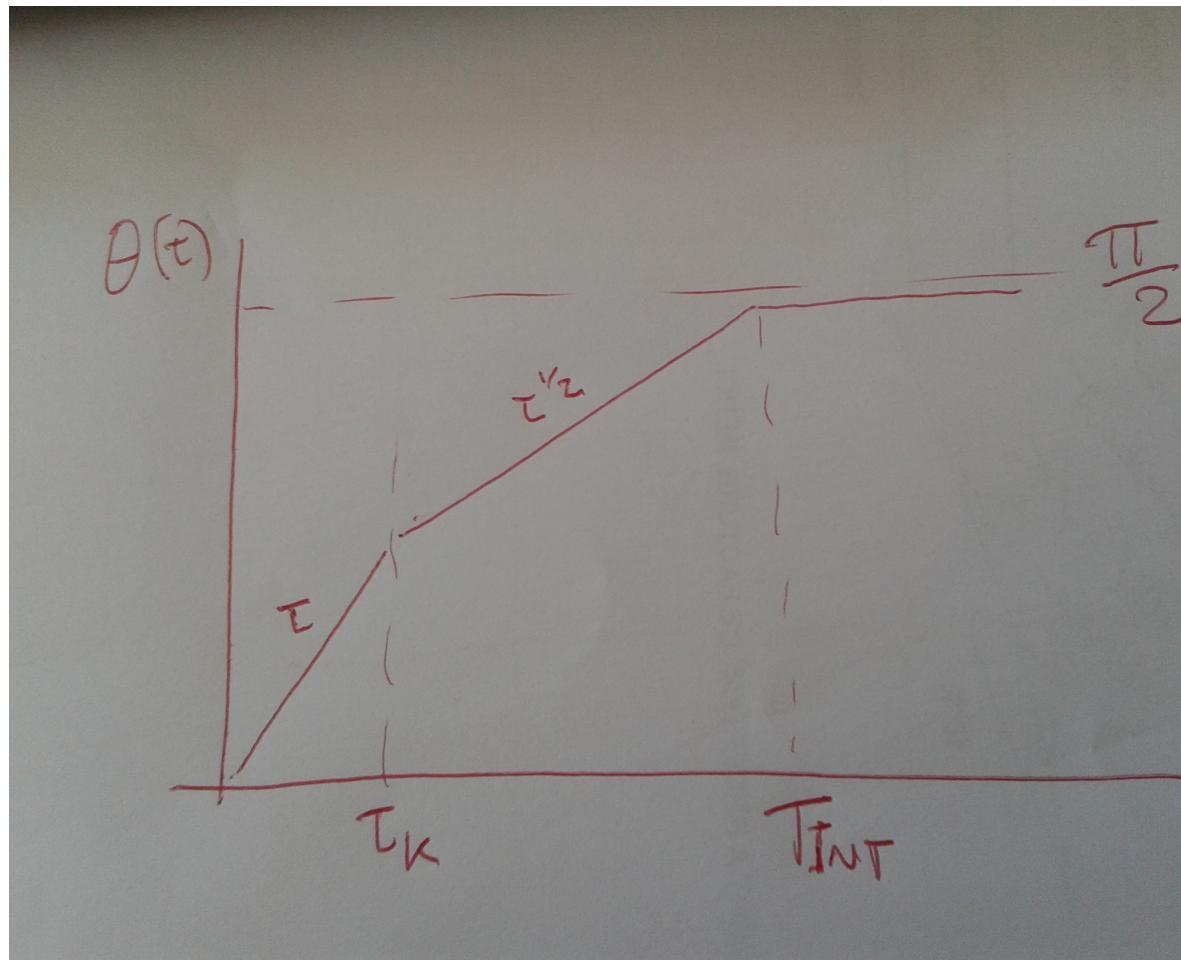
$$\langle \| \langle \mathbf{a}_\perp \rangle_\tau \| \rangle \sim (\epsilon/\tau)^{1/2}. \quad (6)$$

Yielding

$$\begin{aligned} \langle |\Theta(t, \tau)| \rangle &\approx 2\tau \left\langle \frac{\| \langle \mathbf{a}_\perp \rangle_\tau \|}{\| \langle \mathbf{u} \rangle_\tau \|} \right\rangle \\ &\sim \tau^{1/2} \frac{\epsilon^{1/2}}{\langle \| \mathbf{u} \| \rangle} \sim \left( \frac{\tau}{T} \right)^{1/2} \end{aligned} \quad (7)$$

for  $\tau_K \ll \tau \ll T$ . Diffusive scaling directly predictable using Kolmogorov's framework.

We expect :



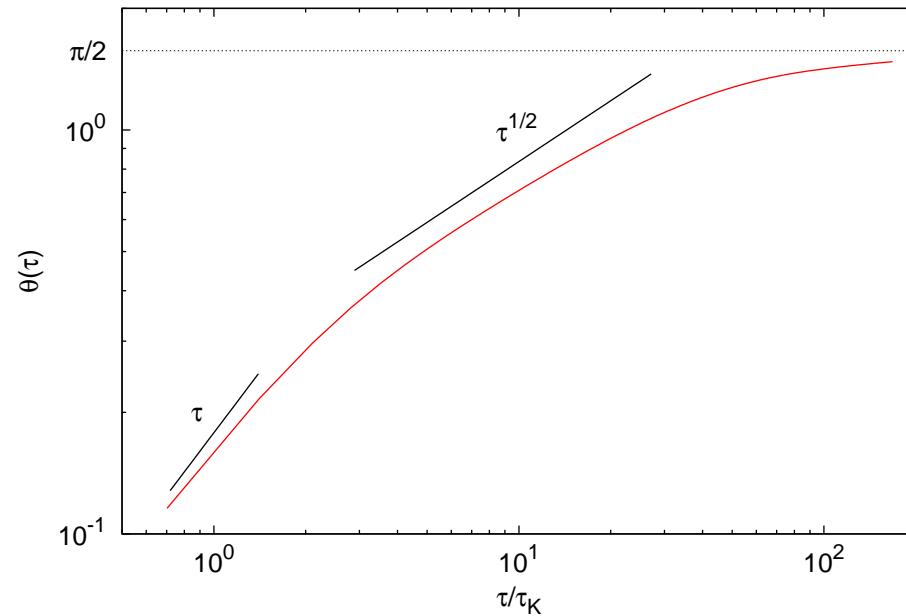
## DNS Database

Thanks to Michael Wilczek & Oliver Kamps

- Forced isotropic incompressible turbulence
- $1024^3$  gridpoints,  $R_\lambda = 225$
- following  $8 \cdot 10^6$  fluid particles during 5.8 integral timescales.

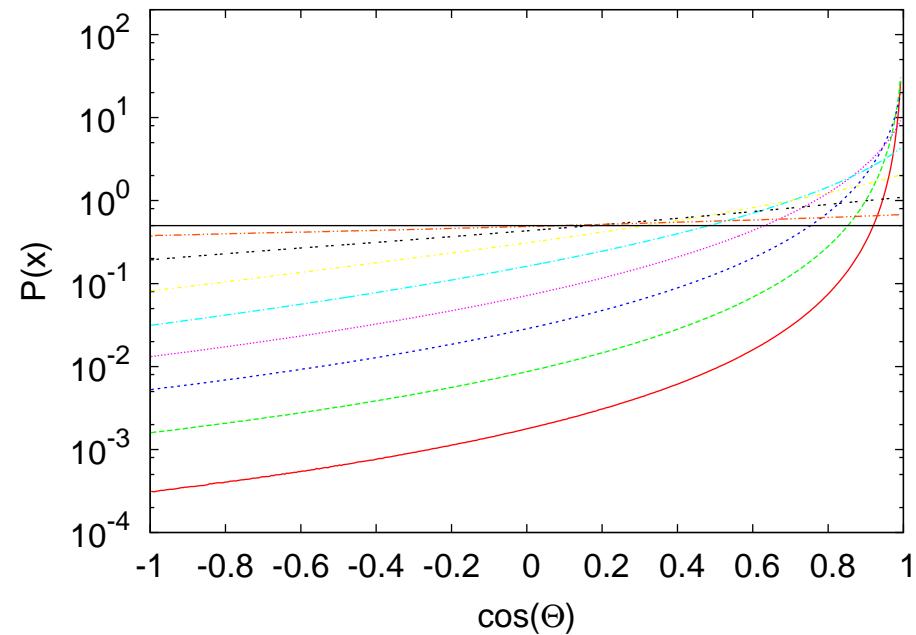
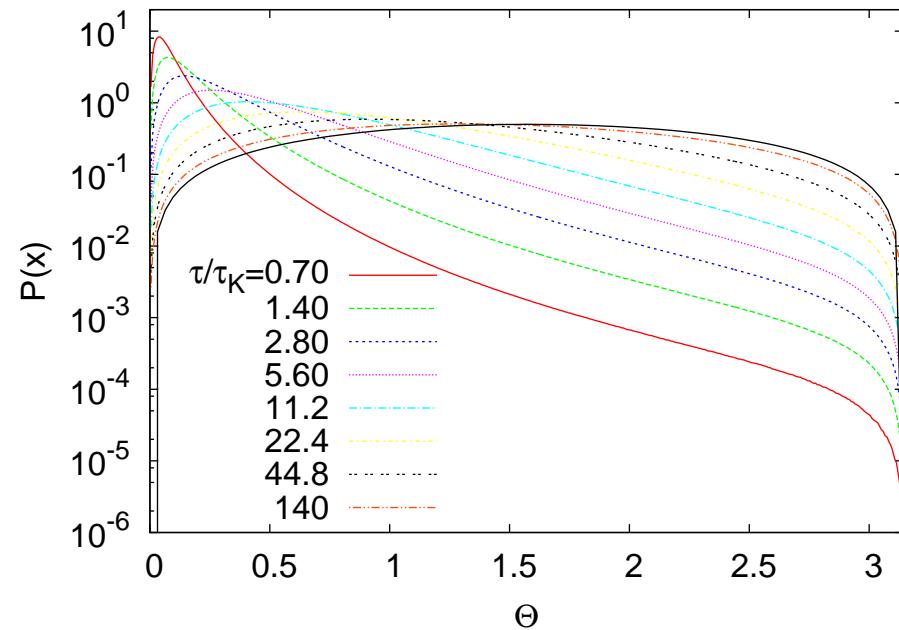
## Behaviour of the mean-coarse-grained-angle

$$\theta(\tau) \equiv \langle |\Theta(t, \tau)| \rangle . \quad (8)$$



Nice !

## Higher order moments, PDFs of $\Theta$



## An analytical model for the PDFs

$$1 - \cos(\Theta(t, \tau)) \approx \frac{1}{2} \Theta(t, \tau)^2 = 2\tau^2 \frac{\sigma_a^2(\tau) \xi_a^2(t, \tau)}{\sigma_u^2(\tau) \xi_u^2(t, \tau)}, \quad (9)$$

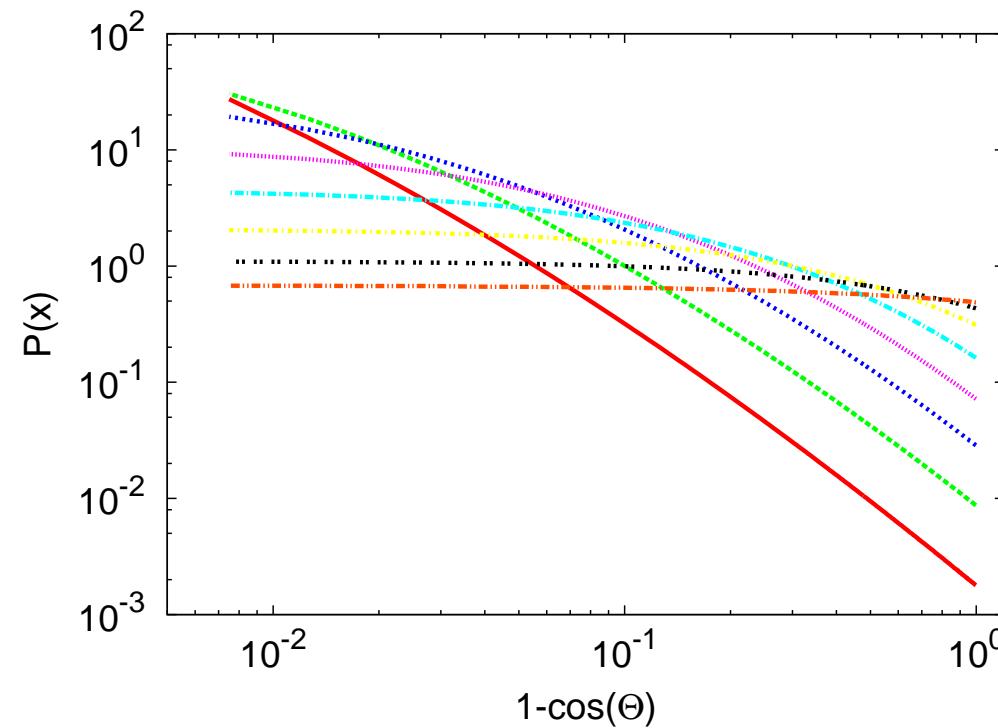
Assuming  $u, a$  independent and Gaussian, small  $\tau, \Theta$ ,

Exact expression for  $P_{1-\cos(\Theta(t, \tau))}$ .

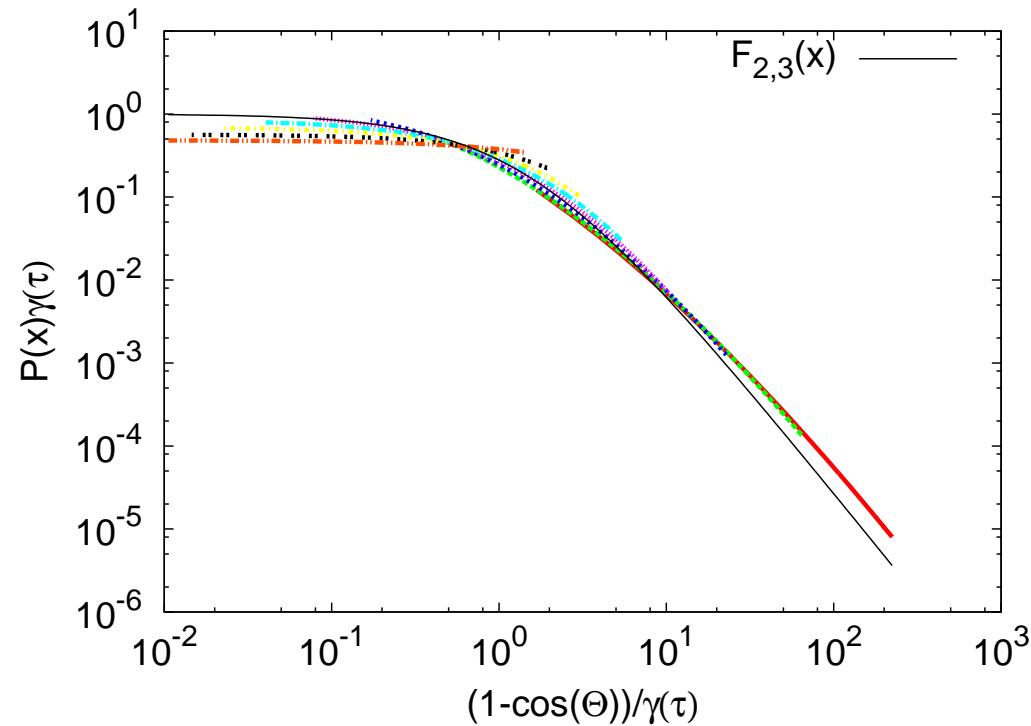
$$\gamma(\tau) P_{1-\cos(\Theta(t, \tau))}(x/\gamma(\tau)) = F_{2,3}(x), \quad (10)$$

where  $\gamma_\tau = \theta(\tau)^2/3$

## Non-normalized PDFs



## Normalized PDFs



$$\text{with } \gamma_\tau = \theta(\tau)^2/3.$$

No adjustable parameters were used to fit the PDF to the  $F$ -distribution !

## Conclusions

1. Coarse grained angular statistics reveal scaling of turbulence
2. The directional change carries the signature of K41 (to a good approximation)
3. PDFs are well fitted to analytical prediction

We only need the particle position at fixed time-intervals, even undersampled, to reveal  
the scaling of the directional change.

Bos, Kadoch, Schneider, Phys. Rev. Lett. (2015)  
hal-01085070