Weak magnetohydrodynamic turbulence and intermittency

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Weak wave turbulence

- Statistical theory of \textit{weakly nonlinear} dispersive waves
- \textbf{Exact solutions} can be found \textit{via} the Zakharov transform

\textbf{2011}

\textit{Wave Turbulence}

\textbf{2013}

\textit{Advances in Wave Turbulence}
Weak MHD turbulence

\[ \frac{\partial z^s}{\partial t} - sb_0 \cdot \nabla z^s = -z^{-s} \cdot \nabla z^s - \nabla P_* \]
\[ s = \pm \quad z^\pm \equiv u \pm b \]

\[ z^s_j(x, t) \equiv \iint \hat{z}^s_j(k, t) e^{ik \cdot x} \, dk = \iint \left[ \epsilon a^s_j(k, t) e^{i\omega_k t} \right] e^{ik \cdot x} \, dk , \]

\[ \frac{\partial a^s_j(k)}{\partial t} = -ie k_m P_{jn} \int_{\mathbb{R}^6} a^{-s}_m(q) a^s_n(p) e^{-i(\omega_k - \omega_p + \omega_q)t} \delta(k - p - q) \, dp \, dq , \]

where \( P_{jn}(k) \equiv \delta_{jn} - k_j k_n / k^2 \) is the projection operator

Resonance condition (3-wave interactions):

\[ \begin{align*}
\omega_k &= \omega_p - \omega_q \\
k &= p + q
\end{align*} \quad \Rightarrow \quad \boxed{q\parallel = 0} \]

No cascade is expected along \( b_0 \)

[Shebalin et al., JPP, 1983]
**Weak turbulence phenomenology**

We assume the balance of turbulence:

\[ z^+ \sim z^- \sim z \]

\[ \lambda_\perp \rightarrow \mathbf{B}_0 \]

Same phenomenology as IK but with **anisotropy**

\[
\tau_{tr} \sim \frac{\tau_{eddy}}{\tau_A} \sim \frac{(\ell_\perp / z_\ell)^2}{\ell_\parallel / b_0} \sim \frac{k_\parallel b_0}{k_\perp^2 z_\ell^2}
\]

We thus obtain:

\[
\varepsilon \sim \frac{z_\ell^2}{\tau_{tr}} \sim \frac{k_\perp^2 z_\ell^4}{k_\parallel b_0} \sim \frac{k_\perp^2 (E(k_\perp, k_\parallel) k_\perp k_\parallel)^2}{k_\parallel b_0} \sim \frac{k_\perp^4 k_\parallel E^2(k_\perp, k_\parallel)}{b_0},
\]

hence the anisotropic (axisymmetric) spectrum:

\[
E^\xi(k_\perp, k_\parallel) \sim \sqrt{\varepsilon b_0} k_\perp^{-2} k_\parallel^{-1/2}
\]

[SG et al., JPP, 2000]
Weak MHD turbulence theory

- Asymptotic equation: \( k_\perp \gg k_\parallel \) is assumed – transverse cascade

\[
\frac{\partial E^\pm(k_\perp, k_\parallel)}{\partial t} = \frac{\pi \varepsilon^2}{B_0} \int_\Delta \cos^2 \phi \sin \theta \frac{k_\perp}{q_\perp} E^\pm(q_\perp, 0) \left[ k_\perp E^\pm(p_\perp, k_\parallel) - p_\perp E^\pm(k_\perp, k_\parallel) \right] dp_\perp dq_\perp
\]

- Spectral solutions:

\[
E^\pm(k_\perp, k_\parallel) = E^\pm(k_\perp)f_\pm(k_\parallel)
\]

\[
E^\pm(k_\perp) \sim k_\perp^{n_\pm}
\]

- Exact solution:

\[
n_+ + n_- = -4
\]

- Nature of the 2D modes; origin of intermittency ??

- Resonance condition

- Direct \( \perp \) cascade is proved

- Condition of locality

[SG et al., JPP, 2000]
Simulations / observations

Zakharov solution is found but an anomalous scaling is obtained in the non-stationary phase
[Thalabard et al., will appear]

Indirect signature in the Jupiter’s magnetosphere
[Saur et al., A&A, 2002]
Direct numerical simulations
Parameters of the numerical experiences

✓ Case A: full equations
\[ \partial_t \zeta^\pm \mp b_0 \partial_y \zeta^\pm + \zeta^\mp \cdot \nabla \zeta^\pm = -\nabla P_0 + \nu_3 \Delta_3 \zeta^\pm \]

✓ Case B: \( u(k_\perp, k_\parallel) = b(k_\perp, k_\parallel) = 0 \) at each time step

| \( nx \times ny \times nz \) | \( E^u_{i=0} = E^b_{i=0} \) | \( |B_0| \) | \( \int_V u \cdot b \, dx \) | \( \nu_3 \) |
|----------------|----------------|------|----------------|------|
| 1536 \times 1536 \times 128 | 0.5 | 20.0 | 0 | \( 4 \times 10^{-15} \) |

✓ TURBO = solver for TURbulent flows with periodic BOundary conditions
Numerical results

\[ \chi^{\pm} = \frac{k_{\perp} z^{\pm}}{k_{\parallel} b_0} < 0.03 \quad \forall \ (k_{\perp}, k_{\parallel}), \quad k_{\parallel} \neq 0 \]

Condition for WT well satisfied

\[ \int_1^{50} E^+(k_{\perp}, k_{\parallel}) dk_{\parallel} \quad \text{(red)} \]

\[ E^-(k_{\perp}, k_{\parallel} = 0) \quad \text{(black)} \]
Spectrogram of the magnetic energy at $k_\perp = 64$

Coherent structures
PDFs of the Elsässer field increments $\delta z^+$

(a) Case A: strong intermittency is found
(b) Case B: almost no intermittency
\[ S_p = \langle (\delta z^+)^{p/2} \rangle \langle (\delta z^-)^{p/2} \rangle = C_p \ell_\perp^\xi(p) \]

A Log-Poisson law is derived:

\[ \zeta(p) = \frac{p}{8} + C_0 - C_0 \left(1 - \frac{3}{4C_0}\right)^{p/2} \]

Co-dimension: \[ C_0 = 1.08 \]

Current sheets

\[ \zeta_p = \frac{p}{8} + 1 - \left(\frac{1}{4}\right)^{p/2} \]

(with \( C_0 = 1 \))
First image of weak MHD turbulence!
Intermittency is found in weak MHD turbulence

This intermittency can be modelled with a log-Poisson law

The 2D modes play a central role via the dissipative structures

Main application: solar/stellar magnetic turbulence