A Bayesian fusion model for reconstruction of high resolution velocity fields in turbulent flows from low resolution measurements

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Turbulent boundary layers at LML

**Space-time resolved data are desired, but:**

- Experiments: limited resolution (PIV: low frequency; Hotwire: point-measurement)
- DNS: low to moderate Reynolds numbers only
Turbulence

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- DNS: low to moderate Reynolds numbers only

Measure and combine different data!
Experiment at LML

S. Coudert et al, *Double large field stereoscopic PIV in a high Reynolds number turbulent boundary layer*. Exp. Fluids, 2011.

Global view

Front view
Objectives

**Input:** $Y(P \times M)$ and $X(Q \times N)$

**Output:** $Z(P \times N)$, $P \gg Q$; $N \gg M$

HTHS: High Temporal High Spatial resolution; HTLS: High Temporal Low Spatial resolution; LTHS: Low Temporal High Spatial resolution
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*HTHS: High Temporal High Spatial resolution; HTLS: High Temporal Low Spatial resolution; LTHS: Low Temporal High Spatial resolution*

Sketch of HTLS and LTHS resolution measurements
Common approaches

Cubic Spline interpolation: mathematical pre-defined interpolation kernel.

Linear Regression: $Z$ as a linear combination of either $X$ or $Y$:

$$Z = B_1 Y \quad \text{or} \quad Z = B_2 X$$

(1)

where coefficients $B_1$ or $B_2$ are learned from $X$ and $Y$. 
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**Limitations**: cannot simultaneously combine all sources of information (correlation)!
Bayesian fusion: Bayes’ theorem

Bayes’ theorem:

\[
p(h|D) = \frac{p(D|h) p(h)}{p(D)}
\]  

where \( h \in H \) is the hypothesis, \( D \) are data (or measurements)

- \( p(h) \): prior probability of \( h \)
- \( p(D) \): prior probability of \( D \)
- \( p(h|D) \): probability of \( h \) given \( D \)
- \( p(D|h) \): probability of \( D \) given \( h \)

Maximum A Posteriori (MAP): find most probable \( h|D \)

\[
h = \arg\max_{h \in H} \{ p(h|D) \} = \arg\max_{h \in H} \{ p(D|h) p(h) \}
\]
Bayesian fusion: The framework

Let \( z, y, x \) be vector forms of \( Z, Y, X \)

**Bayesian fusion:** find \( z| (y, x) \) maximizing *a posteriori* probability:

\[
\hat{z} = \text{argmax}_z \{ p(z|x, y) \}
\]  

(4)

Applying Bayes’ theorem:

\[
p(z|x, y) = \frac{p(x, y|z) p(z)}{p(x, y)} = \frac{p(x|z) p(y|z) p(z)}{p(x, y)}
\]  

(5)

Non-informative prior: \( p(z) \) is constant

\[
\hat{z} = \text{argmax}_z \{ p(x|z) p(y|z) \}
\]  

\[
= \text{argmax}_z \{ p(z|x) p(z|y) \}
\]  

(6)
Bayesian fusion: The models

Models:

\[ z = \mathbb{I}_t x + h_t \quad (\mathbb{I}_t x \perp h_t) \]
\[ z = \mathbb{I}_s y + h_s \quad (\mathbb{I}_s y \perp h_s) \] (7)

Multivariate Gaussian probability:

\[ p(z|x) = \frac{1}{(2\pi)^{NP/2} \left| \Sigma_h \right|^{1/2}} \exp \left\{ -\frac{1}{2} (z - \mathbb{I}_t x)^T \Sigma_h^{-1} (z - \mathbb{I}_t x) \right\} \] (8)

where \( \Sigma_h = h_t h_t^T \) : covariance matrix

Closed-form solution:

\[ \hat{z} = \left( \Sigma_h^{-1} + \Sigma_t^{-1} \right)^{-1} \left( \Sigma_h^{-1} \mathbb{I}_s y + \Sigma_h^{-1} \mathbb{I}_t x \right) \] (9)
Bayesian fusion: Simplified model

**Problem**: estimate and inverse $\Sigma_{hs}$ and $\Sigma_{ht}$ of size $NP \times NP$

$\Rightarrow$ assume diagonal

**Point-wise fusion formula**:

$$\hat{z}(i) = \frac{\sigma_{hs}^2(i)}{\sigma_{hs}^2 + \sigma_{ht}^2(i)} \mathbb{I}_t x(i) + \frac{\sigma_{ht}^2(i)}{\sigma_{hs}^2(i) + \sigma_{ht}^2(i)} \mathbb{I}_s y(i)$$  \hspace{1cm} (10)
Bayesian fusion: Simplified model

Problem: estimate and inverse $\Sigma_{hs}$ and $\Sigma_{ht}$ of size $NP \times NP$
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Point-wise fusion formula:

$$\hat{z}(i) = \frac{\sigma_{hs}^2(i)}{\sigma_{hs}^2 + \sigma_{ht}^2(i)} \mathbb{I}_t x(i) + \frac{\sigma_{ht}^2(i)}{\sigma_{hs}^2(i) + \sigma_{ht}^2(i)} \mathbb{I}_s y(i)$$

(10)

Interpretation: $\hat{z}$ are fused from 2 data sources $\mathbb{I}_t x$ and $\mathbb{I}_s y$ via weighted coefficients $\sigma_{hs}^2$ and $\sigma_{ht}^2$ (learned from $x$ and $y$ only).
Testing database

Streamwise velocity at a plane are extracted from DNS of turbulent channel flow ($Re_\tau = 550$): $P = 10000$, $N = 257 \times 288$. $X$ and $Y$ are virtually subsampled from $Z$. 
Various subsampling ratios

Energy loss:

\[ \Delta \kappa = 1 - \left( \sum_{j \in J} \frac{[Iz_j]^2}{\sum_{j \in J} z_j^2} \right)^{1/2} \]  \hspace{1cm} (11)

<table>
<thead>
<tr>
<th>Case</th>
<th>Ratios</th>
<th>Spacings</th>
<th>Energy loss</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sqrt{N/M}$</td>
<td>$P/Q$</td>
<td>$\Delta z/H$</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
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<tr>
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<td>20</td>
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</tr>
<tr>
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<td>10</td>
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<td>0.11</td>
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<td>4</td>
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<tr>
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<tr>
<td>6</td>
<td>10</td>
<td>10</td>
<td>0.11</td>
</tr>
<tr>
<td>7</td>
<td>20</td>
<td>20</td>
<td>0.22</td>
</tr>
</tbody>
</table>
Results: Normalized Root Mean Square Errors

Normalized Root Mean Square Error (NRMSE):

\[ \epsilon = \left( \frac{\sum_{j \in J} (\hat{z}_j - z_j)^2}{\sum_{j \in J} z_j^2} \right)^{1/2} \]  \hspace{1cm} (12)

Average NRMSE \( \overline{\epsilon} \):

\[ J := \{ (\alpha, \beta, \tau) \} \]

Maximum NRMSE \( \epsilon_{\text{max}} \):

\[ J := \{ (\Delta y/2, \Delta z/2, P\delta t/2Q) \} \]
## Results: NRMSEs (all scales)

<table>
<thead>
<tr>
<th>Case</th>
<th>$\bar{\epsilon}$</th>
<th>$\epsilon_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$I_s y$</td>
<td>$I_t x$</td>
</tr>
<tr>
<td>1</td>
<td>0.14</td>
<td>0.32</td>
</tr>
<tr>
<td>2</td>
<td>0.14</td>
<td>0.54</td>
</tr>
<tr>
<td>3</td>
<td>0.36</td>
<td>0.11</td>
</tr>
<tr>
<td>4</td>
<td>0.68</td>
<td>0.11</td>
</tr>
<tr>
<td>5</td>
<td>0.14</td>
<td>0.11</td>
</tr>
<tr>
<td>6</td>
<td>0.36</td>
<td>0.32</td>
</tr>
<tr>
<td>7</td>
<td>0.68</td>
<td>0.54</td>
</tr>
</tbody>
</table>
## Results: NRMSEs (large scales and small scales)

<table>
<thead>
<tr>
<th>Case</th>
<th>$\bar{\epsilon}$</th>
<th>$\epsilon_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$I_s y$</td>
<td>$I_t x$</td>
</tr>
<tr>
<td><strong>Large scales</strong></td>
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<td></td>
</tr>
<tr>
<td>5</td>
<td>0.08</td>
<td>0.11</td>
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<tr>
<td>6</td>
<td>0.24</td>
<td>0.27</td>
</tr>
<tr>
<td>7</td>
<td>0.55</td>
<td>0.39</td>
</tr>
<tr>
<td><strong>Small scales</strong></td>
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<td></td>
</tr>
<tr>
<td>5</td>
<td>0.98</td>
<td>0.65</td>
</tr>
<tr>
<td>6</td>
<td>0.98</td>
<td>0.89</td>
</tr>
<tr>
<td>7</td>
<td>0.99</td>
<td>0.98</td>
</tr>
</tbody>
</table>
NRMSEs at the most difficult spatial position, $J := \{ (\Delta y/2, \Delta z/2, \tau) \}$. 
Results: NRMSEs in space

NRMSEs at the most difficult instant in time, \( J := \{ (\alpha, \beta, P/2Q) \} \).
Results: a sample snapshot (all scales)
Results: a sample snapshot (large scales)
Results: a sample 1D evolution

Sample evolution of fluctuating streamwise velocity at one of the most difficult position in space ($\Delta y/2, \Delta z/2, \tau$).
Results: a sample 1D evolution

![Graph showing velocity over time with various models and their positions](image-url)

- **Reference**
- **Fusion model**
- **$I_s y$**
- **$I_t x$**
- **Penalized LSE**
- **LTHS positions**
Results: power spectra of streamwise velocity

![Graph showing power spectra with different lines representing Reference, Fusion model, Penalized LSE, and \( I_{t,x} \). The graph plots energy against frequency, showing a -5/3 slope.](image)
Results: power spectra of streamwise velocity
Conclusions and Perspectives

Conclusions

- A Bayesian model using a MAP estimate was developed to reconstruct resolved velocities from sparse measurements.
- A simple point-wise formula is used.
- DNS database is used to study various subsampling ratios.
- The fusion model is superior compared to interpolation or regression, especially in cases where losses of information in space and time are balanced.

Perspectives

- Other possible experimental database.
- More accurate reconstruction methods and other experimental setups.

L.V. Nguyen, J.P. Laval, P. Chainais. A Bayesian fusion model for space-time reconstruction of finely resolved velocities in turbulent flows from low resolution measurements (submitted to J. Stat. Mech.)