# A Bayesian fusion model for reconstruction of high resolution velocity fields in turbulent flows from low resolution measurements

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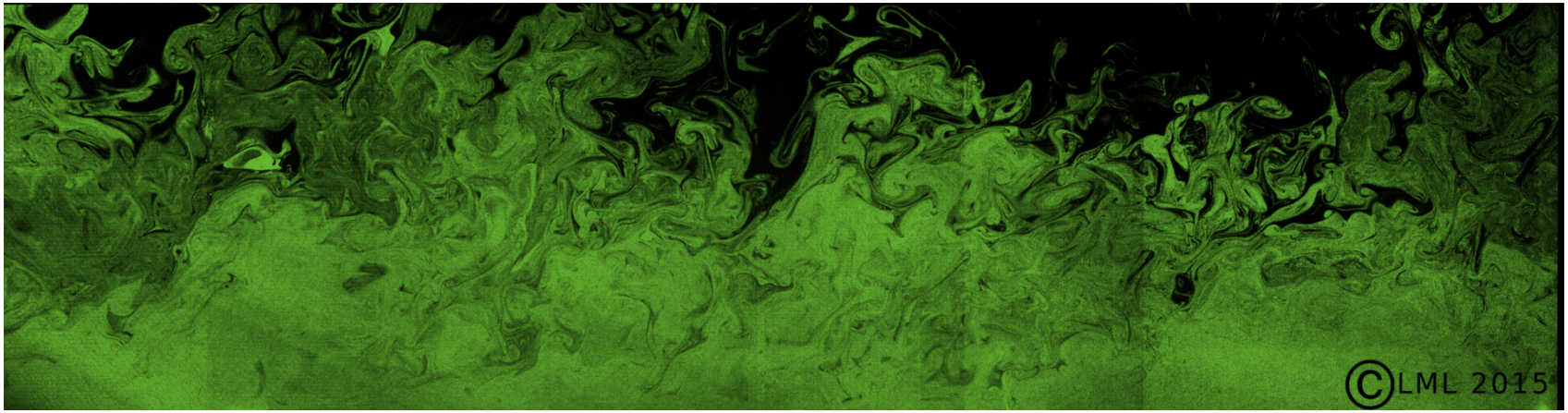
GDR turbulence, June 12, 2015

# Turbulence




*Turbulent boundary layers at LML*

**Space-time resolved data are desired, but:**

- Experiments: limited resolution (PIV: low frequency; Hotwire: point-measurement)
- DNS: low to moderate Reynolds numbers only

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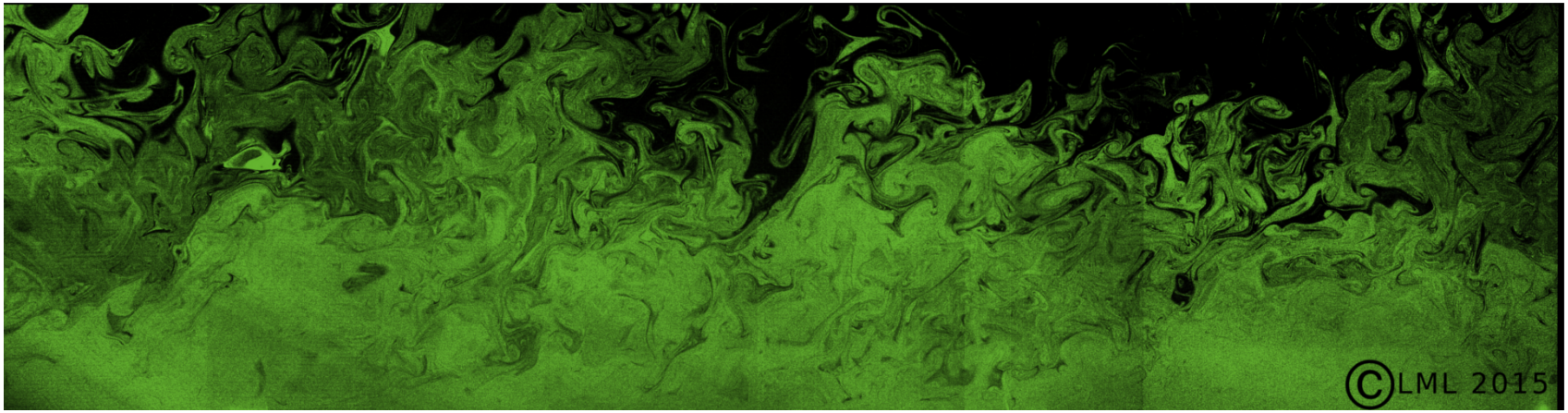


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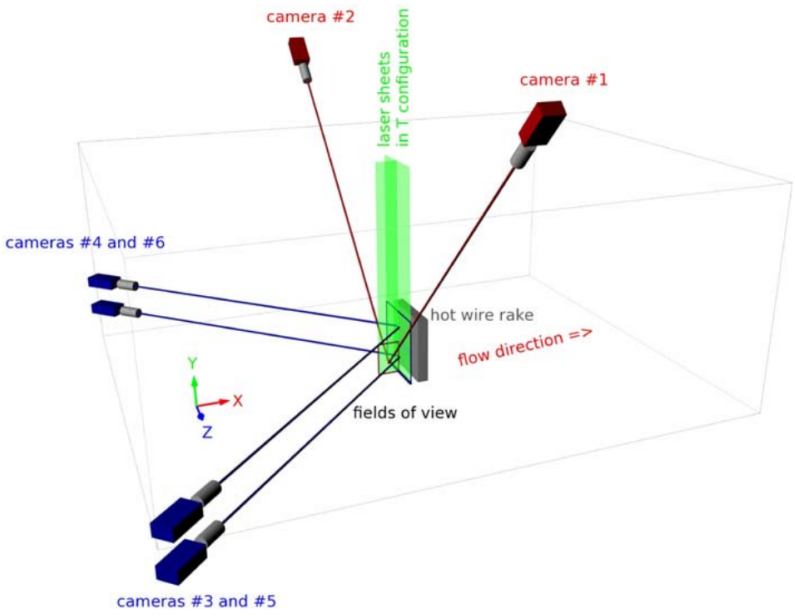
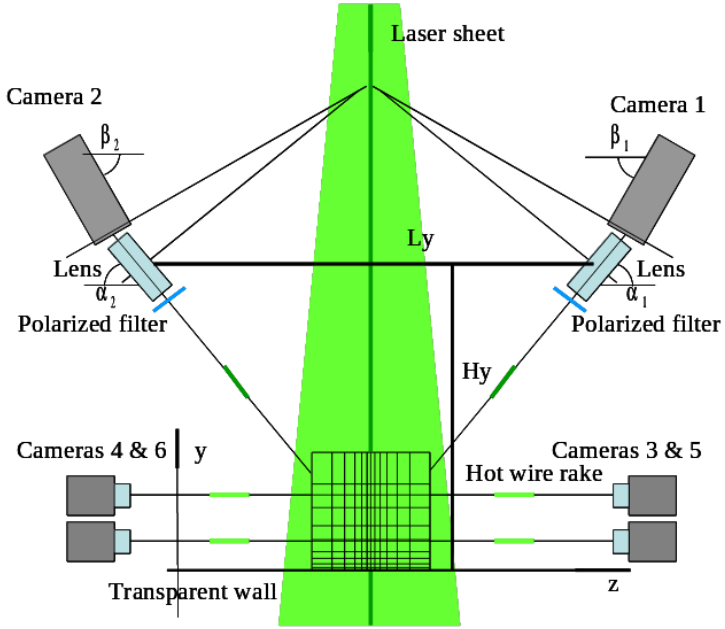
**Measure and combine different data!**

# Experiment at LML

*S. Coudert et al, Double large field stereoscopic PIV in a high Reynolds number turbulent boundary layer. Exp. Fluids, 2011.*




Global view




Front view

# Objectives

**Input**: $Y(P \times M)$ and $X(Q \times N)$
**Output**: $Z(P \times N)$, $P \gg Q$; $N \gg M$

*HTHS: High Temporal High Spatial resolution; HTLS: High Temporal Low Spatial resolution; LTHS: Low Temporal High Spatial resolution*

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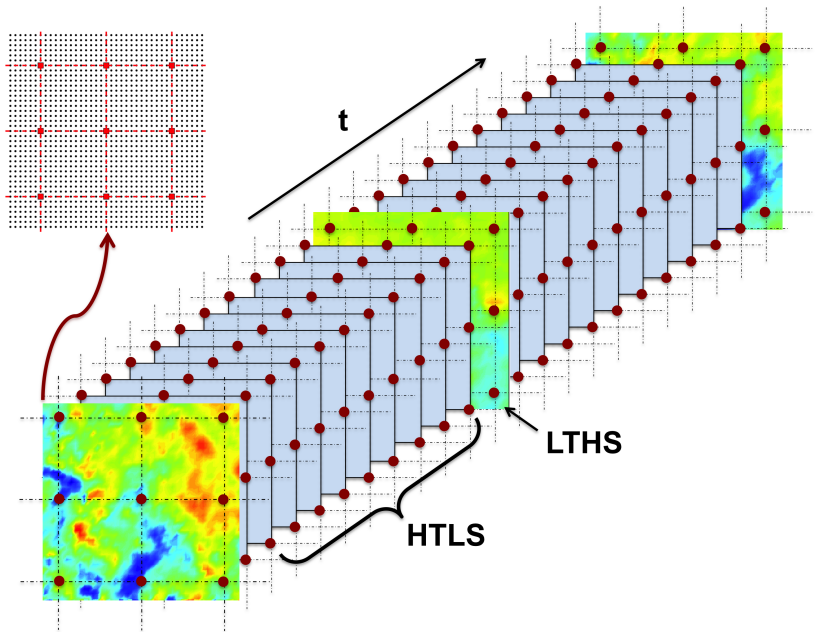
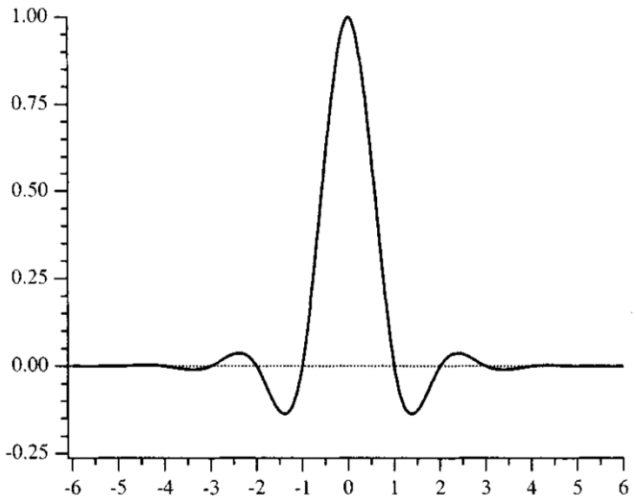



Sketch of HTLS and LTHS resolution measurements

# Common approaches

**Cubic Spline interpolation**: mathematical pre-defined interpolation kernel.




**Linear Regression**: $Z$ as a linear combination of either $X$ or $Y$:

$$Z = B_1 Y \quad \text{or} \quad Z = B_2 X \tag{1}$$

where coefficients $B_1$ or $B_2$ are learned from $X$ and $Y$.

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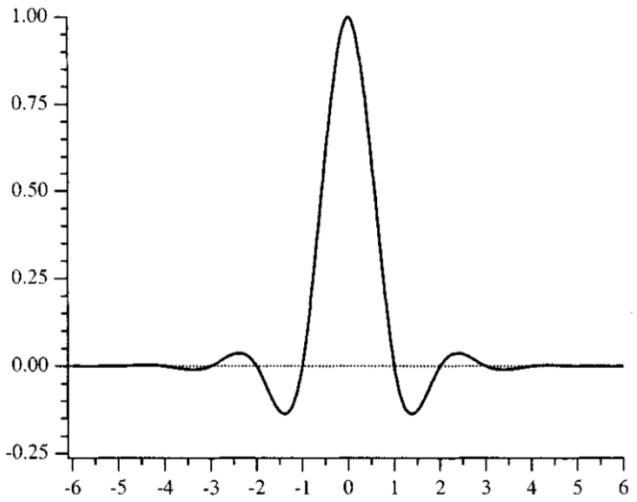


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**Limitations**: cannot simultaneously combine all sources of information (correlation)!

# Bayesian fusion: Bayes' theorem

**Bayes' theorem**:

$$\boxed{p\left(\boldsymbol{h}|\boldsymbol{D}\right) = \frac{p\left(\boldsymbol{D}|\boldsymbol{h}\right)p\left(\boldsymbol{h}\right)}{p\left(\boldsymbol{D}\right)}} \tag{2}$$

where $\boldsymbol{h} \in H$ is the hypothesis, $\boldsymbol{D}$ are data (or measurements)

- $p\left(\boldsymbol{h}\right)$: prior probability of $\boldsymbol{h}$
- $p\left(\boldsymbol{D}\right)$: prior probability of $\boldsymbol{D}$
- $p\left(\boldsymbol{h}|\boldsymbol{D}\right)$: probability of $\boldsymbol{h}$ given $\boldsymbol{D}$
- $p\left(\boldsymbol{D}|\boldsymbol{h}\right)$: probability of $\boldsymbol{D}$ given $\boldsymbol{h}$

**Maximum A Posteriori (MAP)**: find most probable $\boldsymbol{h}|\boldsymbol{D}$

$$\boldsymbol{h} = \underset{\boldsymbol{h}\in H}{argmax}\left\{p\left(\boldsymbol{h}|\boldsymbol{D}\right)\right\} = \underset{\boldsymbol{h}\in H}{argmax}\left\{p\left(\boldsymbol{D}|\boldsymbol{h}\right)p\left(\boldsymbol{h}\right)\right\} \tag{3}$$

# Bayesian fusion: The framework

Let $\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{x}$ be vector forms of $Z, Y, X$

**Bayesian fusion**: find $\boldsymbol{z}|(\boldsymbol{y}, \boldsymbol{x})$ maximizing *a posteriori* probability:

$$\hat{\boldsymbol{z}} = \underset{\boldsymbol{z}}{argmax} \left\{ p\left(\boldsymbol{z}|\boldsymbol{x}, \boldsymbol{y}\right) \right\} \tag{4}$$

**Applying Bayes' theorem**:

$$p\left(\boldsymbol{z}|\boldsymbol{x}, \boldsymbol{y}\right) = \frac{p\left(\boldsymbol{x}, \boldsymbol{y}|\boldsymbol{z}\right) p\left(\boldsymbol{z}\right)}{p\left(\boldsymbol{x}, \boldsymbol{y}\right)} = \frac{p\left(\boldsymbol{x}|\boldsymbol{z}\right) p\left(\boldsymbol{y}|\boldsymbol{z}\right) p\left(\boldsymbol{z}\right)}{p\left(\boldsymbol{x}, \boldsymbol{y}\right)} \tag{5}$$

**Non-informative prior**: $p\left(\boldsymbol{z}\right)$ is constant

$$\begin{aligned} \hat{\boldsymbol{z}} &= \underset{\boldsymbol{z}}{argmax} \left\{ p\left(\boldsymbol{x}|\boldsymbol{z}\right) p\left(\boldsymbol{y}|\boldsymbol{z}\right) \right\} \\ &= \underset{\boldsymbol{z}}{argmax} \left\{ p\left(\boldsymbol{z}|\boldsymbol{x}\right) p\left(\boldsymbol{z}|\boldsymbol{y}\right) \right\} \end{aligned} \tag{6}$$

# Bayesian fusion: The models

**Models**:

$$\begin{aligned} \boldsymbol{z} &= \mathbb{I}_t \boldsymbol{x} + \boldsymbol{h}_t \quad (\mathbb{I}_t \boldsymbol{x} \perp\!\!\!\perp \boldsymbol{h}_t) \\ \boldsymbol{z} &= \mathbb{I}_s \boldsymbol{y} + \boldsymbol{h}_s \quad (\mathbb{I}_s \boldsymbol{y} \perp\!\!\!\perp \boldsymbol{h}_s) \end{aligned} \tag{7}$$

**Multivariate Gaussian probability**:

$$p(\boldsymbol{z}|\boldsymbol{x}) = \frac{1}{(2\pi)^{NP/2} \left|\Sigma_{\boldsymbol{h}_t}\right|^{1/2}} exp\left\{\frac{1}{2}\left(\boldsymbol{z} - \mathbb{I}_t \boldsymbol{x}\right)^T \Sigma_{\boldsymbol{h}_t}^{-1} \left(\boldsymbol{z} - \mathbb{I}_t \boldsymbol{x}\right)\right\} \tag{8}$$

where $\Sigma_{\boldsymbol{h}_t} = \boldsymbol{h}_t \boldsymbol{h}_t^T$: covariance matrix

**Closed-form solution**:

$$\hat{\boldsymbol{z}} = \left(\Sigma_{\boldsymbol{h}_s}^{-1} + \Sigma_{\boldsymbol{h}_t}^{-1}\right)^{-1} \left(\Sigma_{\boldsymbol{h}_s}^{-1} \mathbb{I}_s \boldsymbol{y} + \Sigma_{\boldsymbol{h}_t}^{-1} \mathbb{I}_t \boldsymbol{x}\right) \tag{9}$$

# Bayesian fusion: Simplified model

**Problem**: estimate and inverse $\Sigma_{\boldsymbol{h}_s}$ and $\Sigma_{\boldsymbol{h}_t}$ of size $NP \times NP$
$\Rightarrow$ assume diagonal
**Point-wise fusion formula**:

$$\boxed{\hat{\boldsymbol{z}}(i) = \frac{\sigma^2_{\boldsymbol{h}_s}(i)}{\sigma^2_{\boldsymbol{h}_s} + \sigma^2_{\boldsymbol{h}_t}(i)} \mathbb{I}_t \boldsymbol{x}(i) + \frac{\sigma^2_{\boldsymbol{h}_t}(i)}{\sigma^2_{\boldsymbol{h}_s}(i) + \sigma^2_{\boldsymbol{h}_t}(i)} \mathbb{I}_s \boldsymbol{y}(i)} \quad (10)$$

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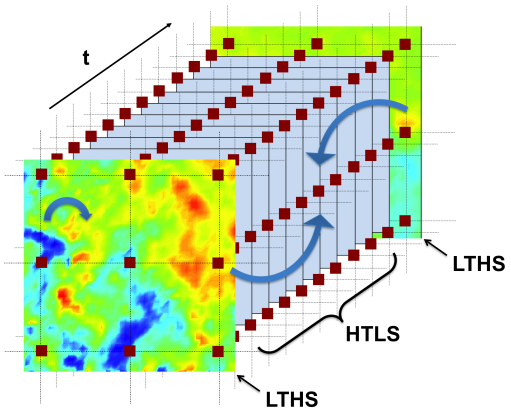
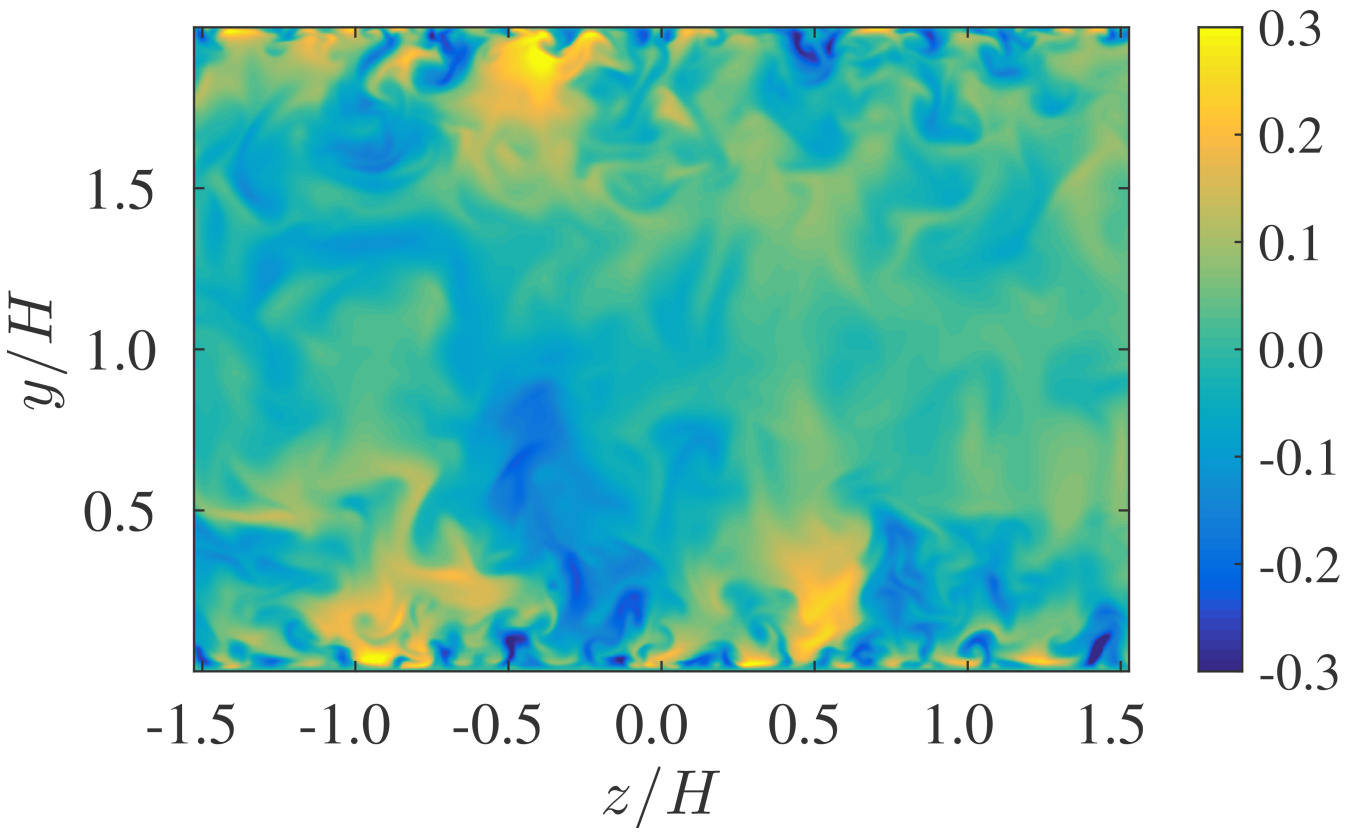
**Interpretation**: $\hat{\boldsymbol{z}}$ are fused from 2 data sources $\mathbb{I}_t \boldsymbol{x}$ and $\mathbb{I}_s \boldsymbol{y}$ via weighted coefficients $\sigma^2_{\boldsymbol{h}_s}$ and $\sigma^2_{\boldsymbol{h}_t}$ (learned from $\boldsymbol{x}$ and $\boldsymbol{y}$ only).



# Testing database

Streamwise velocity at a plane are extracted from DNS of turbulent channel flow ($Re_\tau = 550$): P= 10000, $N = 257 \times 288$.
$X$ and $Y$ are virtually subsampled from Z.



# Various subsampling ratios

**Energy loss**:

$$\Delta\kappa = 1 - \left( \sum_{j \in \mathbb{J}} [\mathbb{L}\boldsymbol{z}]_j^2 / \sum_{j \in \mathbb{J}} \boldsymbol{z}_j^2 \right)^{1/2} \tag{11}$$

| Case | Ratios | | Spacings | | Energy loss | |
|---|---|---|---|---|---|---|
| | $\sqrt{N/M}$ | $P/Q$ | $\Delta z/H$ | $\Delta t\ (s)$ | $\Delta\kappa_s(\%)$ | $\Delta\kappa_t(\%)$ |
| 1 | 5 | 10 | 0.05 | 0.25 | 0.29 | 4.70 |
| 2 | 5 | 20 | 0.05 | 0.50 | 0.29 | 13.12 |
| 3 | 10 | 4 | 0.11 | 0.10 | 5.03 | 0.50 |
| 4 | 20 | 4 | 0.22 | 0.10 | 9.99 | 0.50 |
| 5 | 5 | 4 | 0.05 | 0.10 | 0.29 | 0.50 |
| 6 | 10 | 10 | 0.11 | 0.25 | 5.03 | 4.68 |
| 7 | 20 | 20 | 0.22 | 0.50 | 9.99 | 13.12 |

# Results: Normalized Root Mean Square Errors

Normalized Root Mean Square Error (NRMSE):

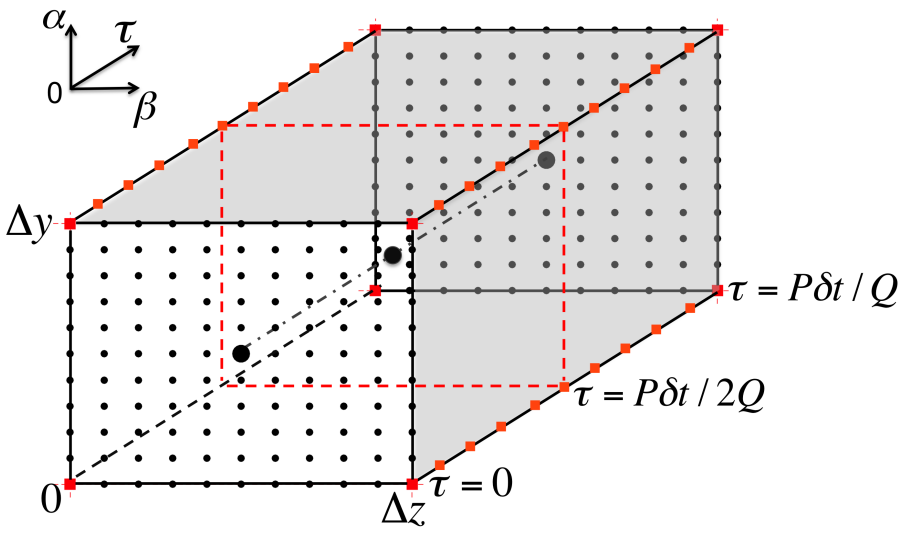
$$\epsilon = \left( \sum_{j \in \mathbb{J}} (\hat{\boldsymbol{z}}_j - \boldsymbol{z}_j)^2 / \sum_{j \in \mathbb{J}} \boldsymbol{z}_j^2 \right)^{1/2} \tag{12}$$

Average NRMSE $\overline{\epsilon}$:

$$\mathbb{J} := \{(\alpha, \beta, \tau)\}$$

Maximum NRMSE $\epsilon_{max}$:

$$\mathbb{J} := \{(\triangle y/2, \triangle z/2, P\delta t/2Q)\}$$



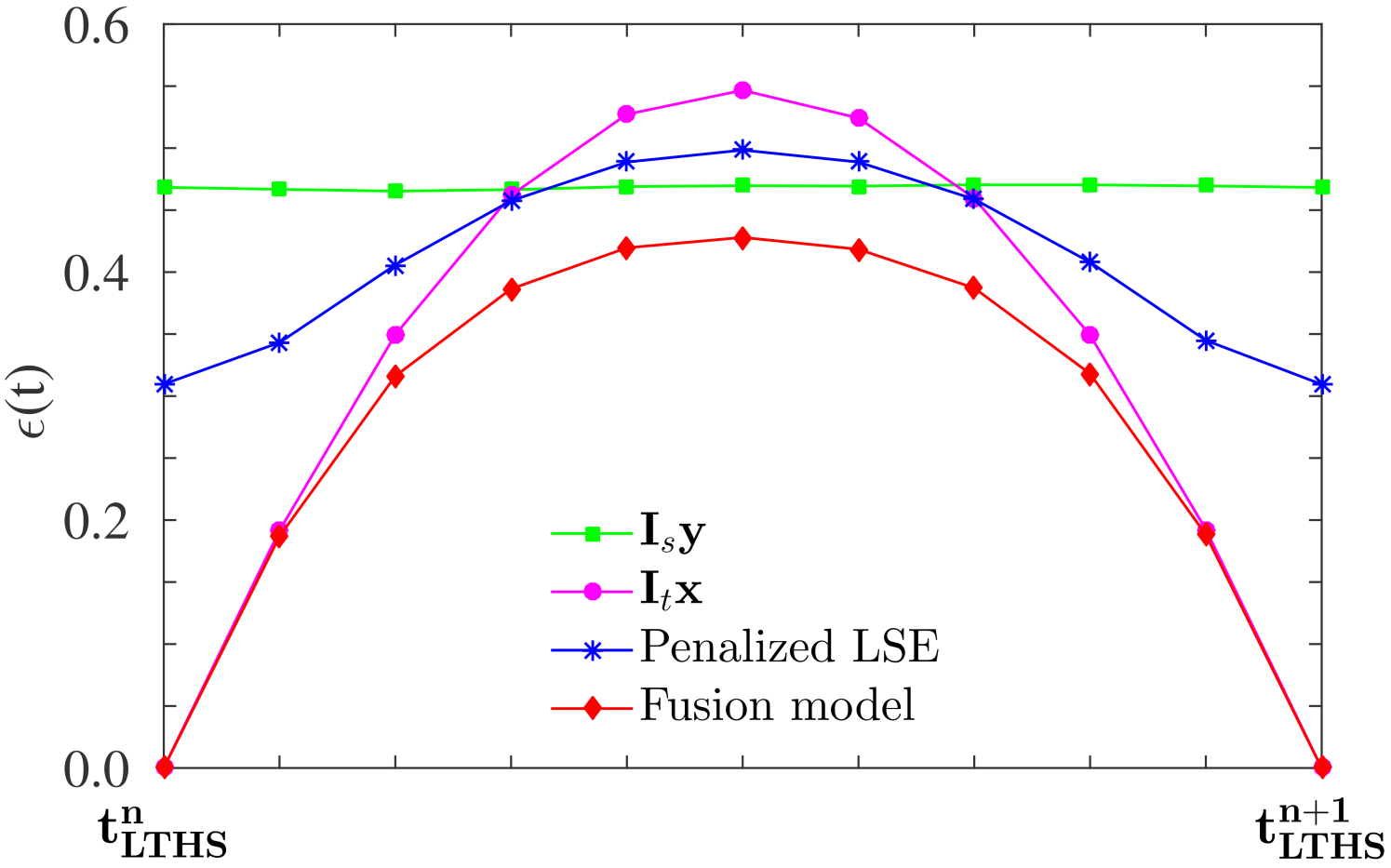
# Results: NRMSEs (all scales)

| Case | $\overline{\epsilon}$ | | | | $\epsilon_{max}$ | | | |
|---|---|---|---|---|---|---|---|---|
| | $\mathbb{I}_s \boldsymbol{y}$ | $\mathbb{I}_t \boldsymbol{x}$ | **LSE** | **Fusion** | $\mathbb{I}_s \boldsymbol{y}$ | $\mathbb{I}_t \boldsymbol{x}$ | **LSE** | **Fusion** |
| 1 | 0.14 | 0.32 | 0.25 | **0.12** | **0.16** | 0.56 | 0.33 | **0.16** |
| 2 | 0.14 | 0.54 | 0.38 | **0.13** | **0.16** | 0.89 | 0.49 | **0.16** |
| 3 | 0.36 | **0.11** | 0.30 | **0.11** | 0.47 | **0.17** | 0.37 | 0.18 |
| 4 | 0.68 | **0.11** | 0.57 | **0.11** | 0.86 | **0.17** | 0.68 | **0.17** |
| 5 | 0.14 | 0.11 | 0.13 | **0.08** | 0.16 | 0.18 | 0.15 | **0.13** |
| 6 | 0.36 | 0.32 | 0.34 | **0.25** | 0.47 | 0.55 | 0.49 | **0.43** |
| 7 | 0.68 | 0.54 | 0.64 | **0.46** | 0.85 | 0.85 | 0.78 | **0.73** |

# Results: NRMSEs (large scales and small scales)

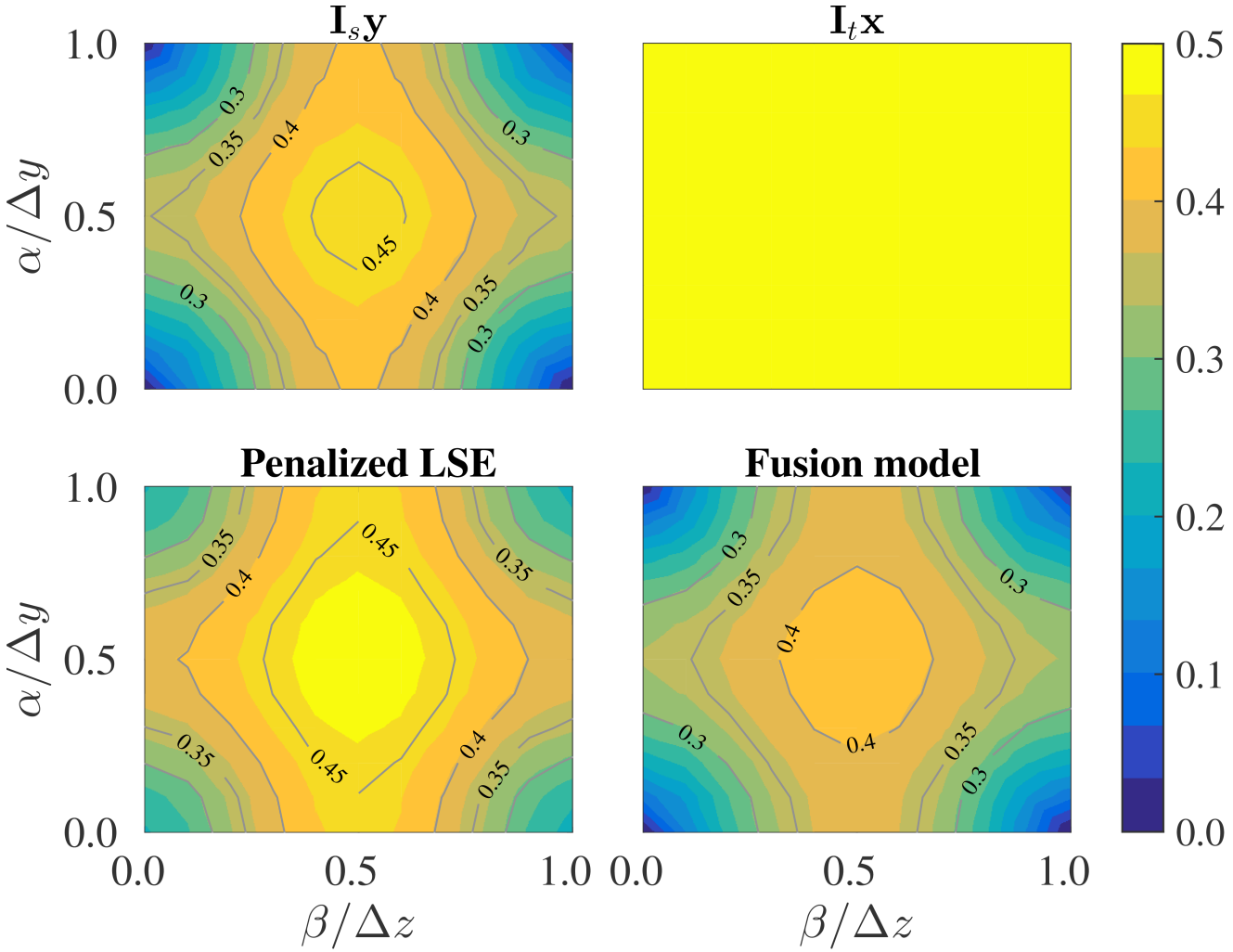
| Case | $\overline{\epsilon}$ | | | | $\epsilon_{max}$ | | | |
|---|---|---|---|---|---|---|---|---|
| | $\mathbb{I}_s \boldsymbol{y}$ | $\mathbb{I}_t \boldsymbol{x}$ | LSE | Fusion | $\mathbb{I}_s \boldsymbol{y}$ | $\mathbb{I}_t \boldsymbol{x}$ | LSE | Fusion |
| **Large scales** | | | | | | | | |
| 5 | 0.08 | 0.11 | 0.10 | **0.06** | **0.06** | 0.15 | 0.11 | **0.06** |
| 6 | 0.24 | 0.27 | 0.24 | **0.16** | 0.22 | 0.43 | 0.27 | **0.21** |
| 7 | 0.55 | 0.39 | 0.51 | **0.31** | 0.57 | 0.62 | 0.57 | **0.44** |
| **Small scales** | | | | | | | | |
| 5 | 0.98 | 0.65 | 0.74 | **0.64** | 0.99 | 0.84 | 0.84 | **0.80** |
| 6 | 0.98 | 0.89 | 0.91 | **0.76** | 0.97 | 1.12 | 1.04 | **0.92** |
| 7 | 0.99 | 0.98 | 0.95 | **0.83** | 0.90 | 1.03 | 0.89 | **0.84** |

# Results: NRMSEs in time



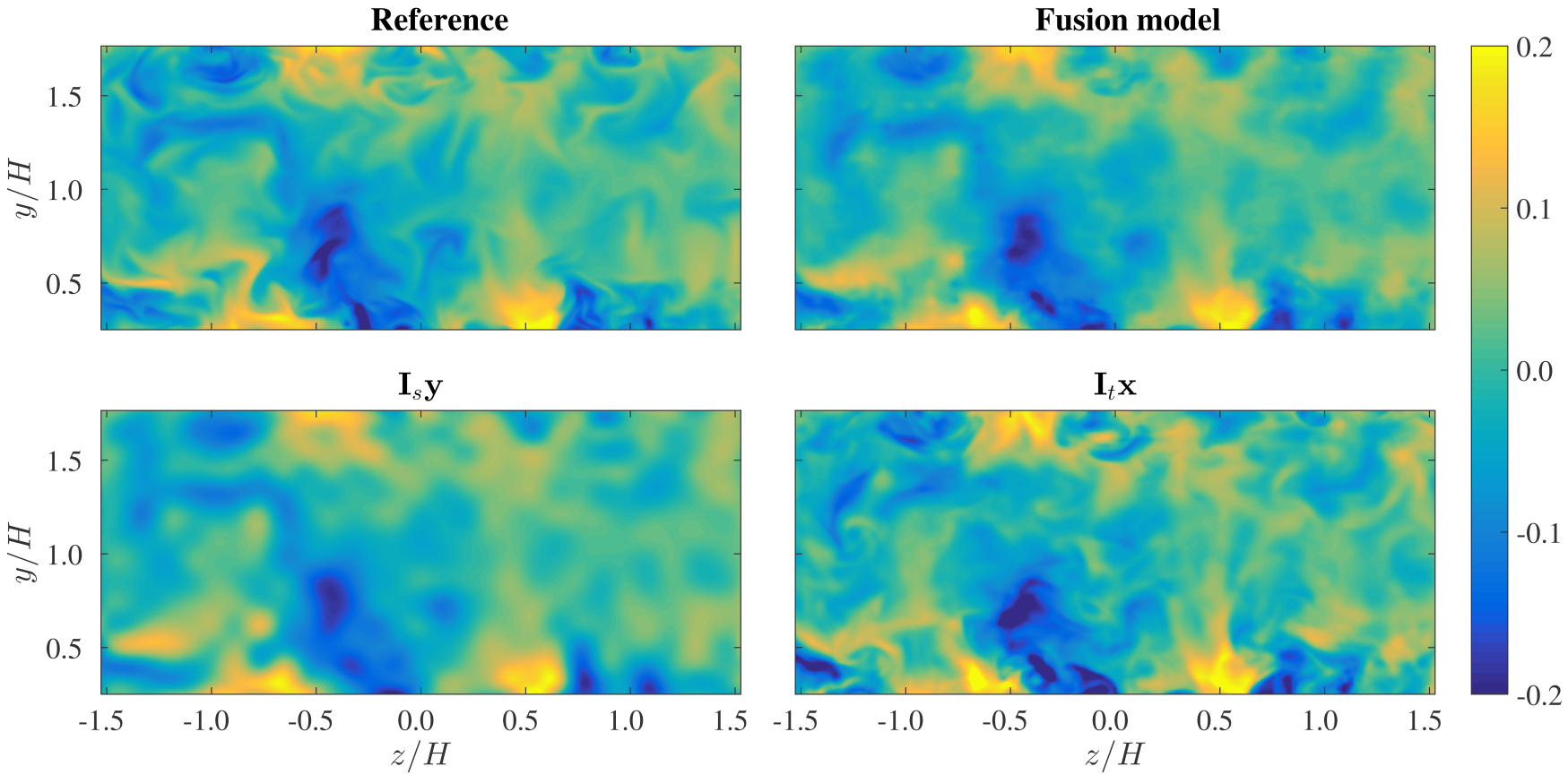

NRMSEs at the most difficult spatial position, $\mathbb{J} := \{(\triangle y/2, \triangle z/2, \tau)\}$.

# Results: NRMSEs in space




NRMSEs at the most difficult instant in time, $\mathbb{J} := \{(\alpha, \beta, P/2Q)\}$.

# Results: a sample snapshot (all scales)



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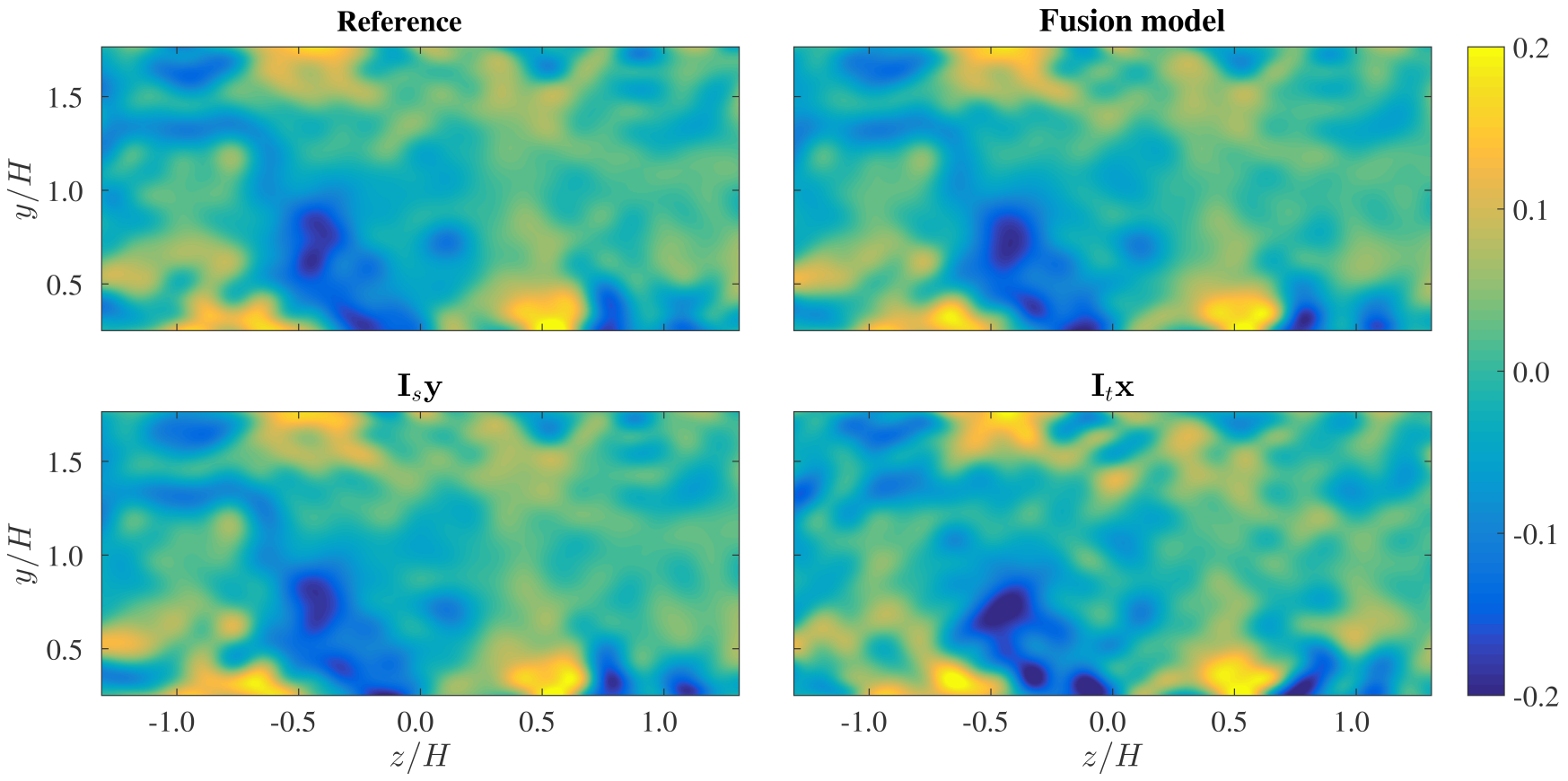
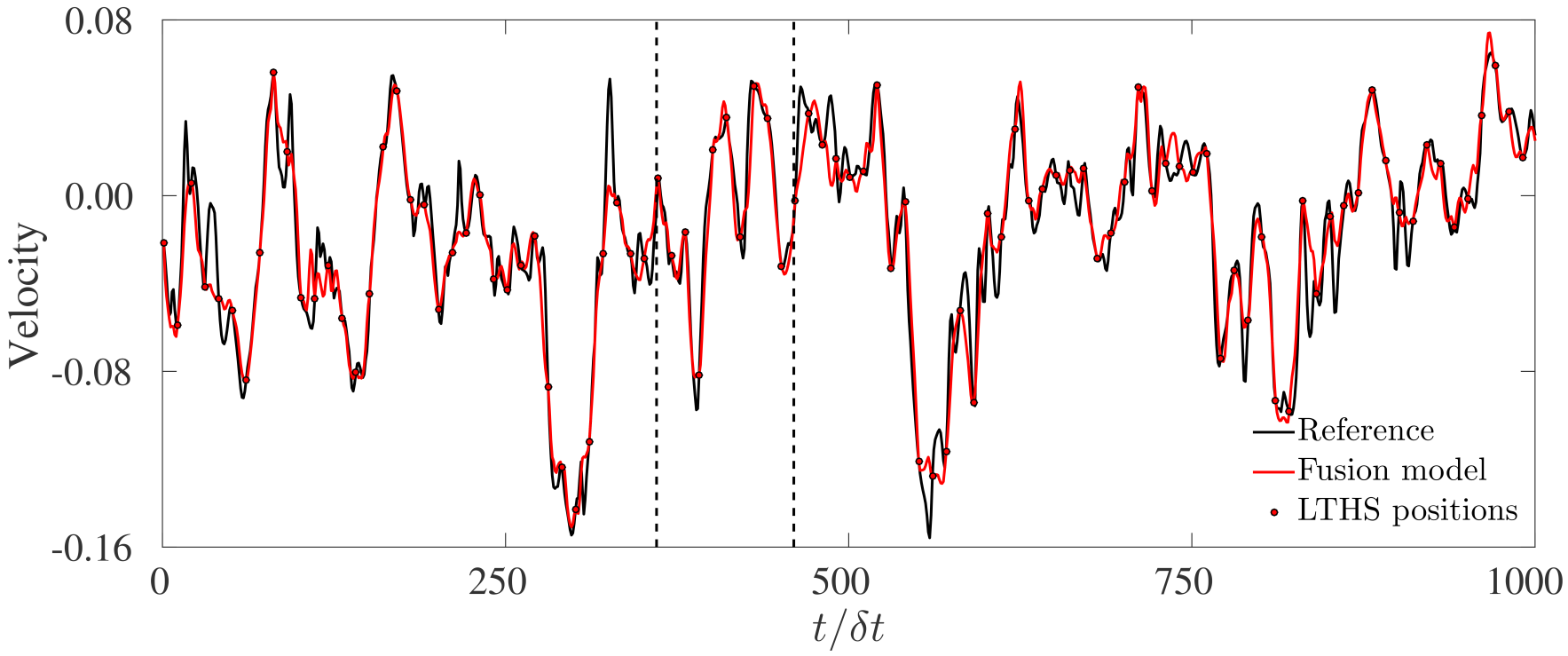
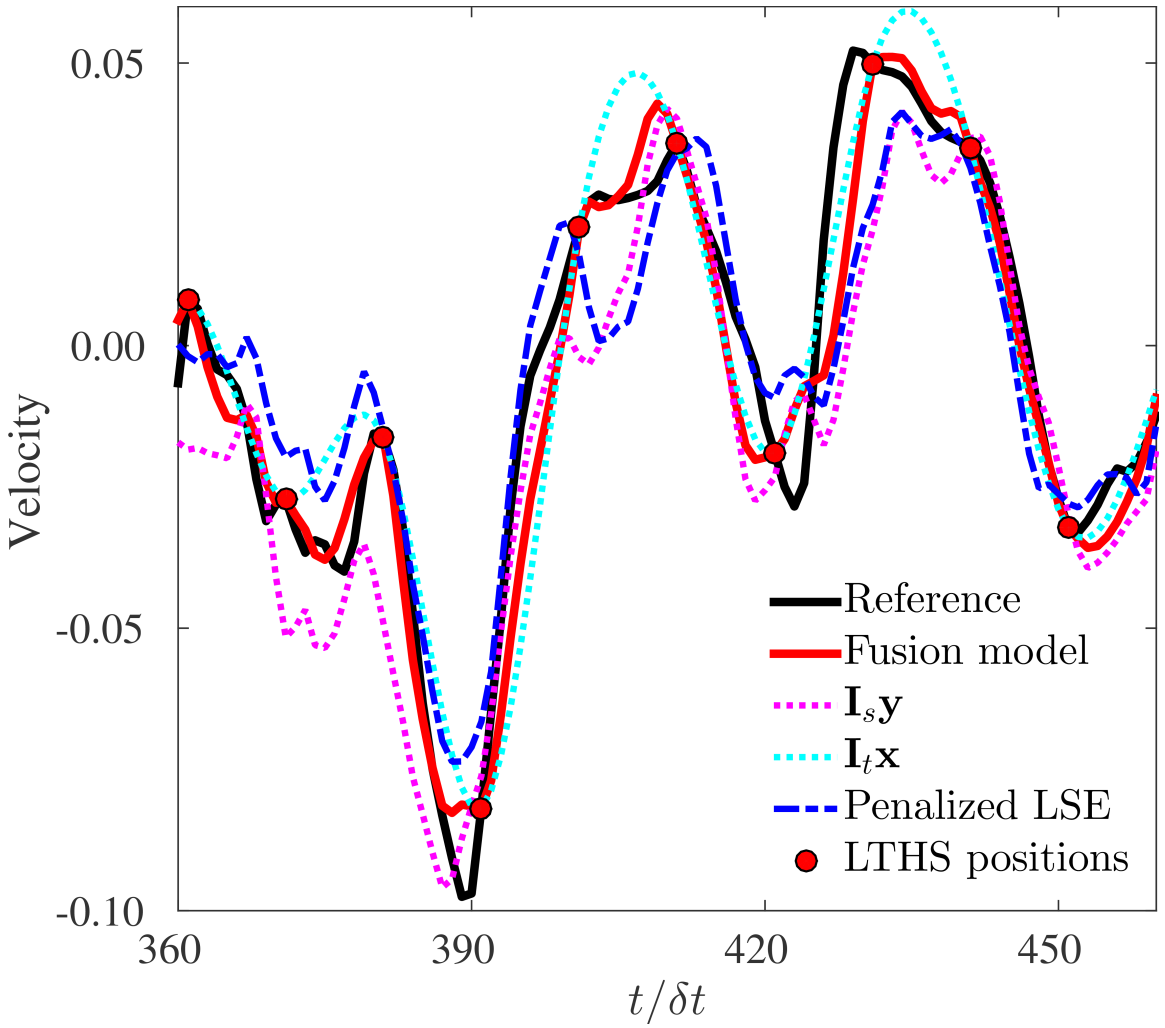


# Results: a sample 1D evolution

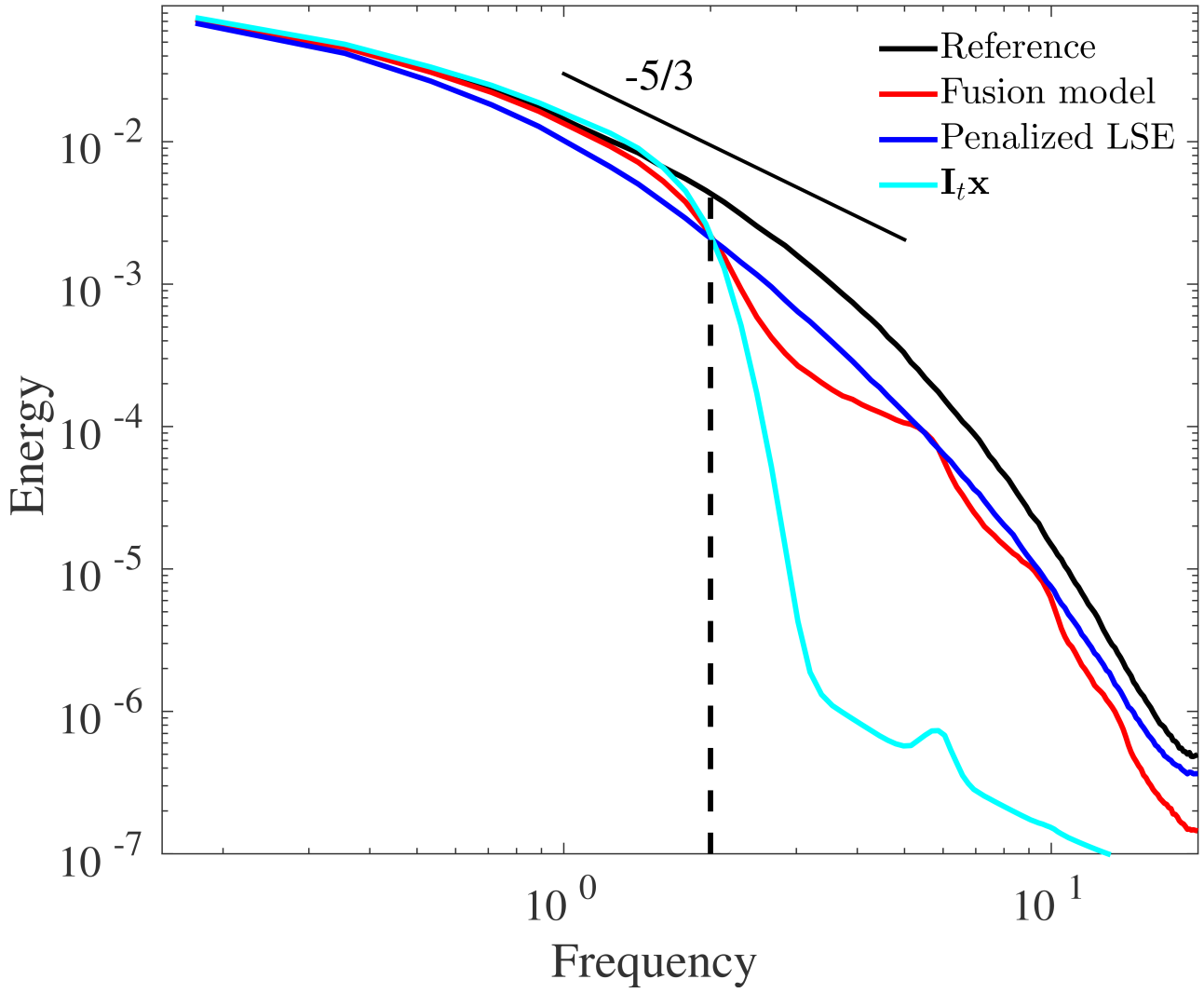



Sample evolution of fluctuating streamwise velocity at of the most difficult position in space $(\triangle y/2, \triangle z/2, \tau)$.

# Results: a sample 1D evolution



# Results: power spectra of streamwise velocity



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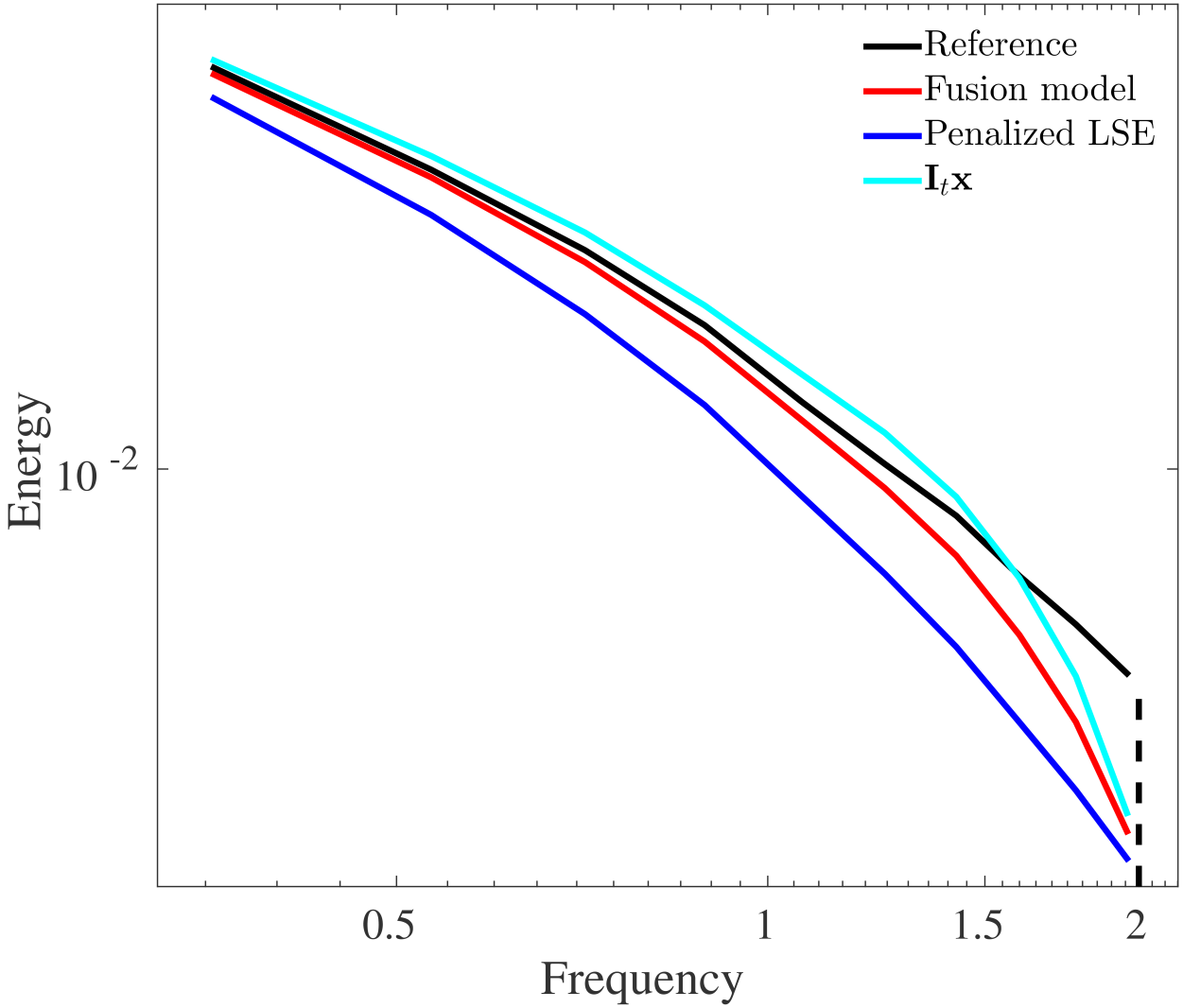

# Conclusions and Perspectives

## Conclusions

- A Bayesian model using a MAP estimate was developed to reconstruct resolved velocities from sparse measurements
- A simple point-wise formula is used
- DNS database is used to study various subsampling ratios
- The fusion model is superior compared to interpolation or regression, especially in cases where losses of information in space and time are balanced.

## Perspectives

- Other possible experimental database
- More accurate reconstruction methods and other experimental setups

*L.V. Nguyen, J.P. Laval, P. Chainais. A Bayesian fusion model for space-time reconstruction of finely resolved velocities in turbulent flows from low resolution measurements (submitted to J. Stat. Mech.)*