Méthode de Galerkin discontinue pour la simulation des écoulements turbulents

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GDR Turbulence, Poitiers, 2012
Outline

Introduction

Numerical method
   Modal DG discretization

Taylor-Green vortex
   The Taylor-Green vortex flow
   DNS computations

Vortex dipole-wall collision
   Case description
   Computations

Conclusion
Context

Discontinuous Galerkin (DG) methods

- Combine features of FEM and FVM
- Unstructured meshes (multi-elements, accurate representation of boundaries)
- High-order accuracy (polynomial approximation)
- Compact stencil, advantageous for parallelization based on MPI
- Natural framework for multiscale turbulence modelling
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Modal DG discretization

- **3D compressible Navier Stokes equations**

\[
\frac{\partial \mathbf{w}}{\partial t} + \nabla \cdot \mathbf{F} = 0, \quad \mathbf{F} = \mathbf{F}^{\text{conv}} (\mathbf{w}) - \mathbf{F}^{\text{visc}} (\mathbf{w}, \nabla \mathbf{w})
\]

- **Spatial discretization**

\[
\Omega \quad \rightarrow \quad \Omega_h
\]

\[
\Omega_h = \bigcup_{j=1}^{N} \kappa_j
\]

- **Polynomial expansion in the elements**

\[
\mathbf{w}_h (\mathbf{x}, t) = \sum_{k=1}^{N_p} \mathbf{W}_k (t) \phi_k (\mathbf{x}), \quad \phi \in \mathcal{V}_h^p, \quad \text{dim} (\mathcal{V}_h^p) = N_p
\]
Modal DG discretization

- Discontinuous solution across element’s faces

- Numerical flux for convective terms, e.g. Local Lax-Friedrichs
- Numerical flux with lifting operators (BR2\textsuperscript{1}) for viscous terms
- The BR2 scheme is compact
- RK3 scheme for the integration in time of the discrete system

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- Model for 3D incompressible freely decaying turbulence in a periodic box
- Analytical initialization in physical space

\[
\begin{align*}
u_0(x, y, z) & = V_0 \sin \left(\frac{x}{L}\right) \cos \left(\frac{y}{L}\right) \cos \left(\frac{z}{L}\right) \\
v_0(x, y, z) & = -V_0 \cos \left(\frac{x}{L}\right) \sin \left(\frac{y}{L}\right) \cos \left(\frac{z}{L}\right) \\
w_0(x, y, z) & = 0 \\
p_0(x, y, z) & = p_\infty + \frac{\rho_0}{16} \left(\cos\left(\frac{2x}{L}\right) + \cos\left(\frac{2x}{L}\right)\right) \left(\cos\left(\frac{2z}{L}\right) + 2\right)
\end{align*}
\]

- \( Re = V_0 L / \nu \), \( M = 0.1 \) for all computations
- Monitoring the evolution of the mean energy and enstrophy

\[
E_k = \frac{1}{V} \int_V \frac{u \cdot u}{2} d\mathbf{x} \quad \zeta = \frac{1}{V} \int_V \frac{\omega \cdot \omega}{2} d\mathbf{x}
\]
The Taylor-Green vortex flow

Illustration of the evolution of the flow

\[ t = 0 \quad t = 1 \quad t = 3 \quad t = 4.5 \]

\[ t = 5.5 \quad t = 7 \quad t = 8 \quad t = 11 \]

Isosurfaces of \(|\omega|\), Fourier spectral DNS computation (Code provided by R. Cant, Cambridge Univ.)
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RE=500 : Details of the computations

<table>
<thead>
<tr>
<th>Computation</th>
<th>Method</th>
<th>Accuracy (p+1)</th>
<th>Nb. DOFs/Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>DG 48p1</td>
<td>DG</td>
<td>2</td>
<td>$96^3 / 48^3$</td>
</tr>
<tr>
<td>DG 32p2</td>
<td>DG</td>
<td>3</td>
<td>$96^3 / 32^3$</td>
</tr>
<tr>
<td>DG 24p3</td>
<td>DG</td>
<td>4</td>
<td>$96^3 / 24^3$</td>
</tr>
<tr>
<td>DG 16p5</td>
<td>DG</td>
<td>6</td>
<td>$96^3 / 16^3$</td>
</tr>
<tr>
<td>DG 12p7</td>
<td>DG</td>
<td>8</td>
<td>$96^3 / 12^3$</td>
</tr>
<tr>
<td>PS 128</td>
<td>PS</td>
<td>-</td>
<td>$128^3$</td>
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</table>

Key points

- Evolution of the mean kinetic energy and enstrophy
- Representation of the energy spectrum
RE=500 : Statistics and spectra

Evolution of energy

Evolution of enstrophy

Energy spectra at t=9
DNS computations

RE=1600 : Details of the computations

<table>
<thead>
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<tbody>
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<tr>
<td>DG 64p3</td>
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<tr>
<td>DG 96p3</td>
<td>DG</td>
<td>4</td>
<td>384^3 / 96^3</td>
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<tr>
<td>PS 256</td>
<td>PS</td>
<td>-</td>
<td>256^3</td>
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</tbody>
</table>

Key points

- Physical representation of vortices
- Behaviour of under-resolved DG computations
RE=1600 : Statistics and spectra

Evolution of energy

Evolution of enstrophy

Energy spectra at t=9
RE=1600 : Flow structure

- Modulus of vorticity on the $(y, z)$ plane at $x = 0$
Taylor-Green: summary

- $Re = 500$
  - High-order DG simulations show similar results to PS computations at iso-number of DOFs
  - Representation of a richer spectral content as the polynomial order is increased

- $Re = 1600$
  - Accurate spectral and physical representation of turbulent structures for well-resolved DG computations
  - Acceptable results for under-resolved computations
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- 2D wall bounded flow involving complex wall-vortex interactions
- Analytical initialization of two counter rotating vortex monopoles in a square domain

\[ \omega_i = \pm \omega_e \left( 1 - \left( \frac{r_i}{r_0} \right)^2 \right) e^{-\left( \frac{r_i}{r_0} \right)^2}, \quad i = 1, 2 \]

- \( Re = h \sqrt{2E_0/\nu} = 1000 \), where \( h \) is half of the domain length and \( E_0 \) the initial mean kinetic energy
- Monitoring the evolution of the mean energy and enstrophy

\[ E_k = \frac{1}{V} \int_V \frac{u \cdot u}{2} d\mathbf{x} \quad \zeta = \frac{1}{V} \int_V \frac{\omega^2}{2} d\mathbf{x} \]
Illustration of the evolution of the flow\(^3\)

\[
t = 0 \hspace{2cm} t = 0.2 \hspace{2cm} t = 0.4 \\
\]

\[
t = 0.6 \hspace{2cm} t = 0.8 \hspace{2cm} t = 1.2 \\
\]

\(^3\)Isosurfaces of $\omega_z$, Spectral DNS (Keetels et al., JCP, 2007)
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Mesh convergence: detail of computations

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<th>Computation</th>
<th>Elements</th>
<th>Nb. DOFs</th>
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<td>DG 128p2</td>
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<td>$384^2$</td>
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<tr>
<td>DG 256p2</td>
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<tr>
<td>Fourier-Chebychev</td>
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<td>$2048 \times 1024$</td>
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</table>

- Key points
  - Evolution of the mean energy and enstrophy, vorticity contours
  - Representation of the collision process
Mesh convergence : Evolution of energy and enstrophy
Mesh convergence: Vorticity plots

$t = 0.5$

$t = 0.6$

$t = 0.8$
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**Stretched meshes**

<table>
<thead>
<tr>
<th>Computation</th>
<th>Order</th>
<th>Mesh</th>
<th>Nb. Dofs</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>DG (p5)</td>
<td>6</td>
<td>(32 \times 16)</td>
<td>18432</td>
</tr>
</tbody>
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Evolution of energy and enstrophy

![Graph showing energy and enstrophy evolution over time](image-url)
Vorticity plots

\[ t = 0.5 \]

\[ t = 0.6 \]

\[ t = 0.8 \]
Summary

- The DG method is able to represent complex near-wall phenomena
- High-order DG allows for a drastic reduction of the number of DOFs for the representation of the collision process
Conclusion

- DG methods well adapted to DNS of decaying turbulent flows
  - Good representation of structures in physical space
  - Accurate description of the spectral content
  - Good prediction of mean quantities
  - Equivalent to pseudo-spectral methods in terms of accuracy

- DG methods show an interest for wall bounded flows
  - Representation of complex near-wall phenomena