Doak's momentum potential theory of energy flux used to study a solenoidal wavepacket

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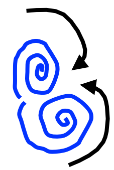
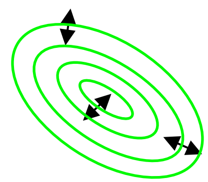
Acoustic and other motions

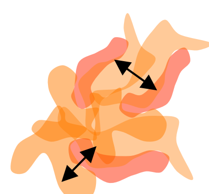
$$\mathcal{L}(u_s) = 0$$



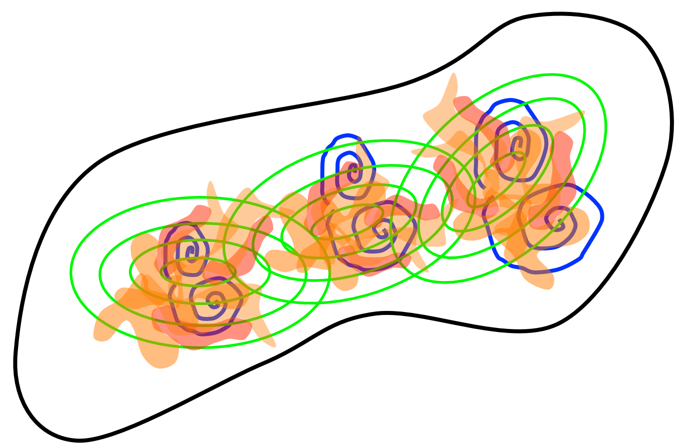
$$\mathcal{L}(p) = 0$$



$$\mathcal{L}(\sigma) = 0$$



Rayleigh (1877)
Chu & Kovaznay (1958)
Cantrell & Hart (1964)
Morfey (1971)
Doak (1974,1989)
Pierce (1981)
Jenvey (1989)
Myers (1994)
Goldstein (2003,2005)



Doak's momentum-potential theory of energy flux

Linear momentum density: sum of solenoidal and irrotational components

$$\rho v_i \equiv B_i - \partial \psi / \partial x_i, \qquad \partial B_i / \partial x_i \equiv 0,$$

Mass conservation is then a linear equation:

$$\partial \rho / \partial t - \partial^2 \psi / \partial x_i^2 = 0.$$

Assume time-stationarity in mass density fluctuations:

$$\partial \rho' / \partial t - \partial^2 \psi' / \partial x_i^2 = 0, \qquad \partial^2 \bar{\psi} / \partial x_i^2 = 0$$

And so, for any time-stationary motion

$$\rho v_i = \overline{\rho v_i} + (\rho v_i)' = \bar{B}_i(x_k) + B_i'(x_k, t) - \partial \psi'(x_k, t) / \partial x_i,$$

$$\partial \bar{B}_i / \partial x_i = \partial B_i' / \partial x_i = 0.$$

Doak's momentum-potential theory of energy flux

From the energy conservation equation:

$$\frac{\partial}{\partial x_j}\left(\overline{H}\;\overline{B}_j + \overline{H'B'_j} - \overline{H'\frac{\partial \psi'}{\partial x_j}}\right) = \overline{E_{S_i}}$$

where H is the total enthalpy.

For fluctuating quantities only:

$$\frac{\partial}{\partial x_j}\left(\overline{H'B'_j} - \overline{H'\frac{\partial \psi'}{\partial x_j}}\right) = \overline{\frac{\partial \psi'}{\partial x_i}(\vec{\Omega} \wedge \vec{u})'_i} - \overline{B'_i(\vec{\Omega} \wedge \vec{u})'_i} + \overline{E_{S_i}}.$$

cf. Moyal (1952); Lighthill (1952); Chu & Kovaznay (1958).

A model problem: wavepacket forcing with vorticity enhancement

$$\vec{u} = (u + \epsilon u_\epsilon, v + \epsilon v_\epsilon)$$

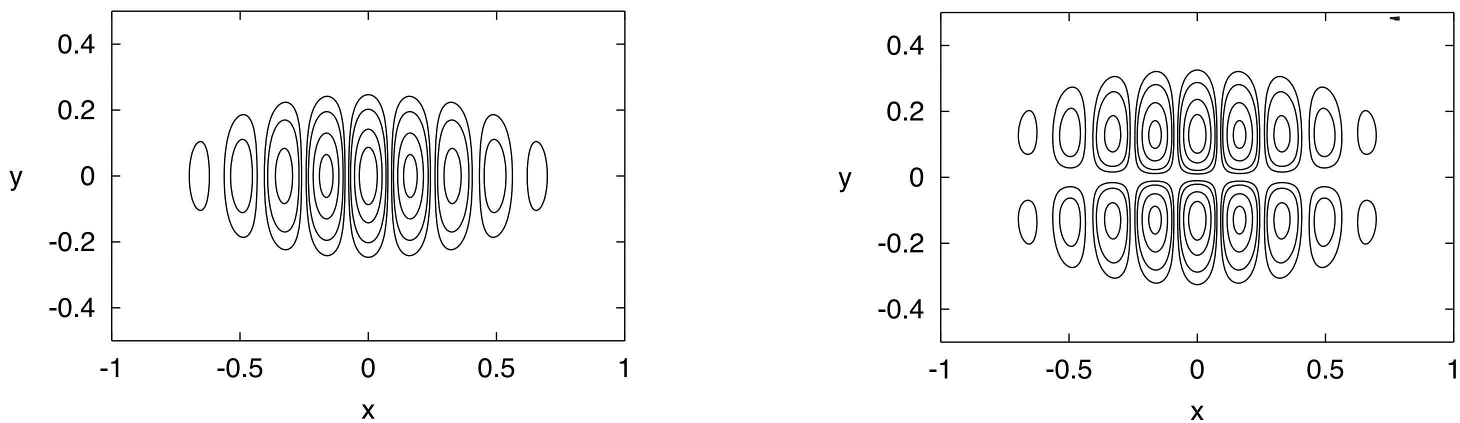
$$u = \frac{\partial \phi}{\partial x}; v = \frac{\partial \phi}{\partial y}$$
$$u_\epsilon = -\frac{\partial \psi}{\partial y}; v_\epsilon = \frac{\partial \psi}{\partial x},$$

$$\phi = A \exp\left(-\frac{(x - x_o)^2}{\lambda_x^2} - \frac{(y - y_o)^2}{\lambda_y^2}\right) \cos(k_x x - k_x U_c t),$$
$$\psi = y\phi,$$

A model problem: wavepacket forcing with vorticity enhancement

$(A = 0.01, \epsilon = 10)$

$\nabla \cdot \vec{u}$ $\nabla \wedge \vec{u}$



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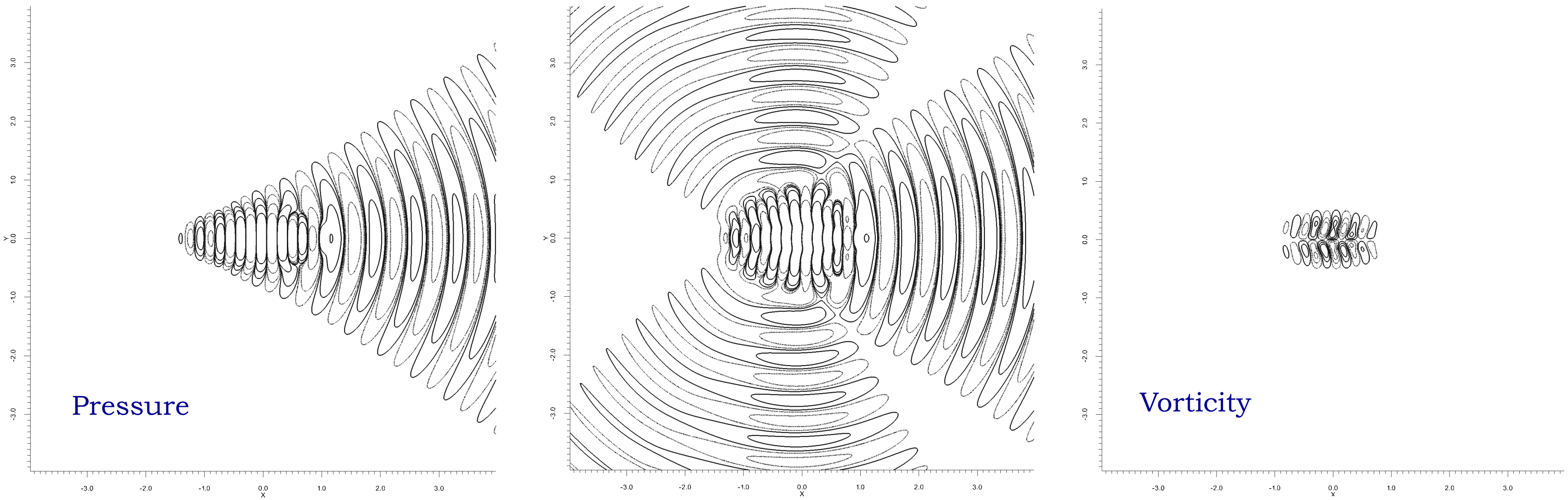
A model problem: flux and source terms

$$\frac{\partial}{\partial x_j}\left(\overline{H'B'_j} - \overline{H'\frac{\partial\psi'}{\partial x_j}}\right) = \overline{\frac{\partial\psi'}{\partial x_i}(\vec{\Omega}\wedge\vec{u})'_i} - \overline{B'_i(\vec{\Omega}\wedge\vec{u})'_i} + \overline{E_{S_i}}.$$

$$\overline{E_{S_i}} = \overline{\rho\mathbf{u}\cdot\mathbf{a}_f} = \overline{(\overline{B}_i + B'_i + \frac{\partial\psi'}{\partial x_i})a_i} + \overline{(\overline{B}_i + B'_i + \frac{\partial\psi'}{\partial x_i})\epsilon a^{\epsilon}_i}$$

A model problem: response of 2D Euler equations

$(A = 0.01, \epsilon = 0)$ $(A = 0.01, \epsilon = 10)$

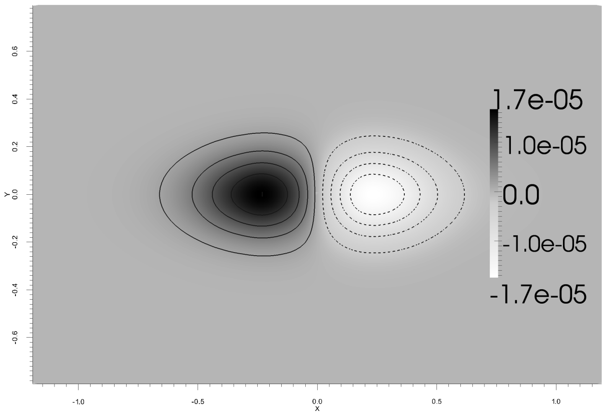


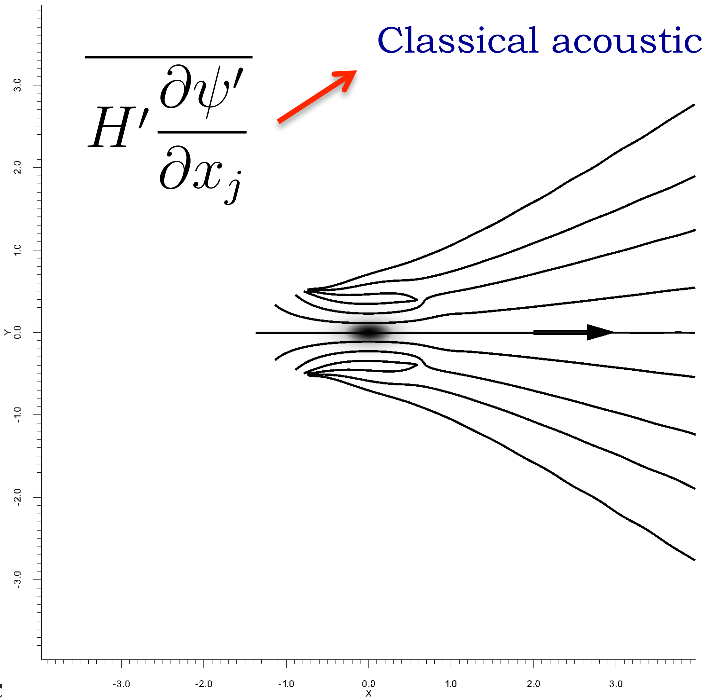
$$\overline{E_{S_i}} = \overline{\rho \mathbf{u} \cdot \mathbf{a}_f} = \overline{(\overline{B}_i + B_i' + \frac{\partial \psi'}{\partial x_i}) a_i} + \overline{(\overline{B}_i + B_i' + \frac{\partial \psi'}{\partial x_i}) \epsilon a_i^{\epsilon}}$$

Wavepacket without vorticity enhancement: source and flux terms

$$(A = 0.01, \epsilon = 0)$$

$$\frac{\partial}{\partial x_j}\left(\overline{H'B'_j} - \overline{H'\frac{\partial \psi'}{\partial x_j}}\right) = \overline{\frac{\partial \psi'}{\partial x_i}(\vec{\Omega} \wedge \vec{u})'_i} - \overline{B'_i(\vec{\Omega} \wedge \vec{u})'_i} + \overline{E_{S_i}}.$$





$$\overline{H'\frac{\partial \psi'}{\partial x_j}}$$

$$\overline{E_{S_i}} = \overline{(\overline{B}_i + B'_i + \frac{\partial \psi'}{\partial x_i})a_i} + \overline{(\overline{B}_i + B'_i + \frac{\partial \psi'}{\partial x_i})\epsilon a^{\epsilon}_i}.$$

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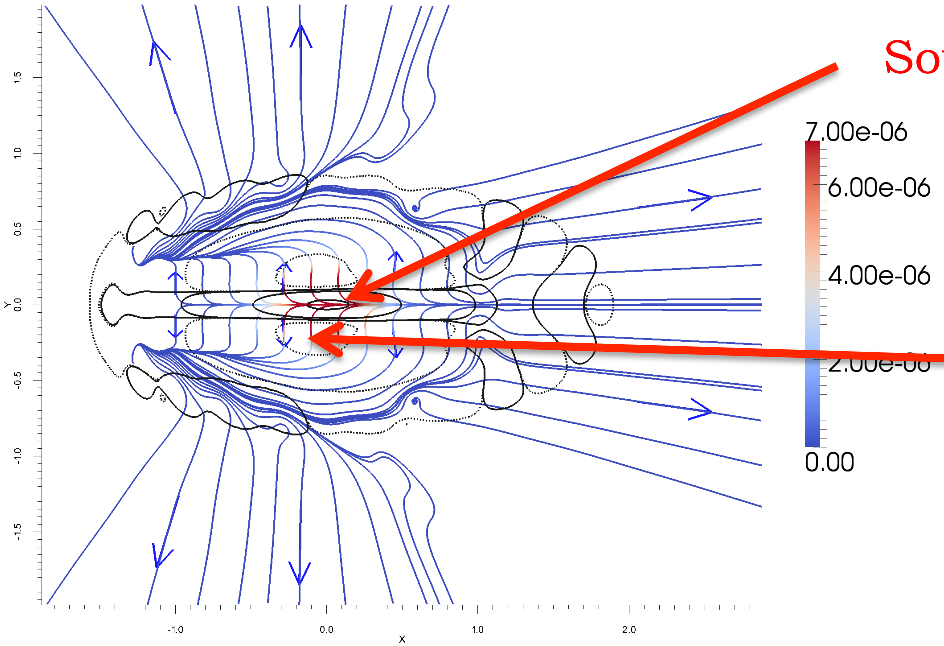


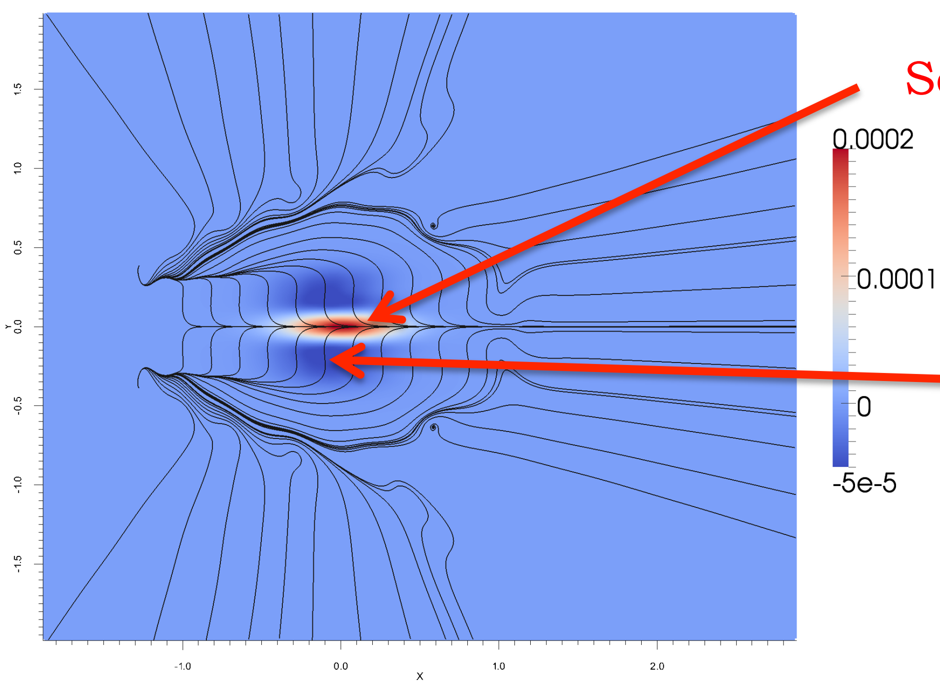
Source: work *performed on* fluid by forcing.

Sink: work *removed from* fluid by forcing

Wavepacket with vorticity enhancement: source and flux terms

$$(A = 0.01, \epsilon = 10)$$

$$\frac{\partial}{\partial x_j}\left(\overline{H'B'_j} - \overline{H'\frac{\partial \psi'}{\partial x_j}}\right) = \overline{\frac{\partial \psi'}{\partial x_i}(\vec{\Omega} \wedge \vec{u})'_i} - \overline{B'_i(\vec{\Omega} \wedge \vec{u})'_i} + \overline{E_{S_i}}.$$



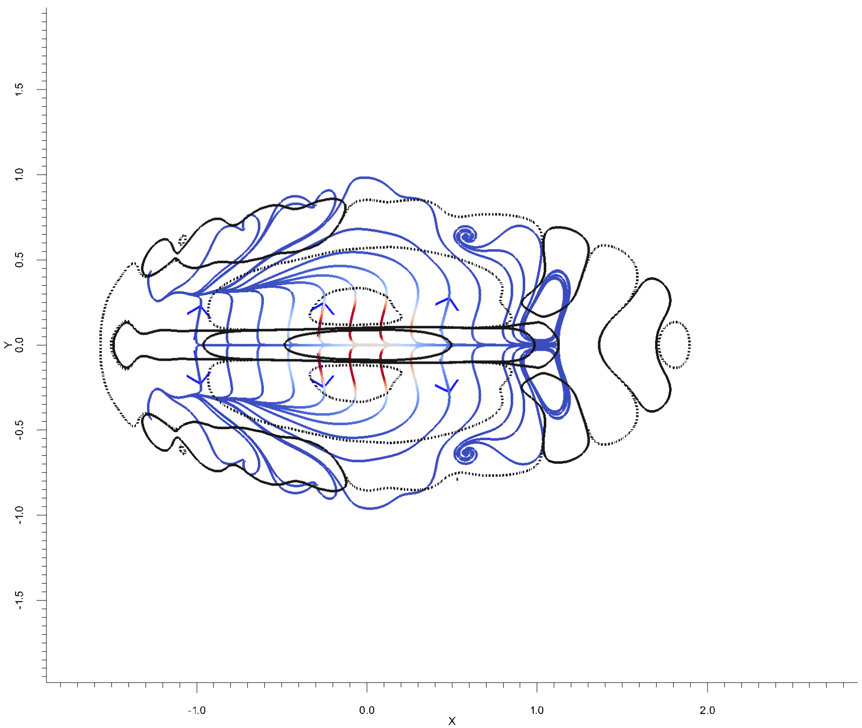
Source: work *performed on* fluid by forcing.

Sink: work *removed from* fluid by forcing

Flux due to solenoidal momentum fluctuations

$(A = 0.01, \epsilon = 10)$

$$\frac{\partial}{\partial x_j}\left(\overline{H'B'_j} - \overline{H'\frac{\partial \psi'}{\partial x_j}}\right) = \overline{\frac{\partial \psi'}{\partial x_i}(\vec{\Omega} \wedge \vec{u})'_i} - \overline{B'_i(\vec{\Omega} \wedge \vec{u})'_i} + \overline{E_{S_i}}.$$



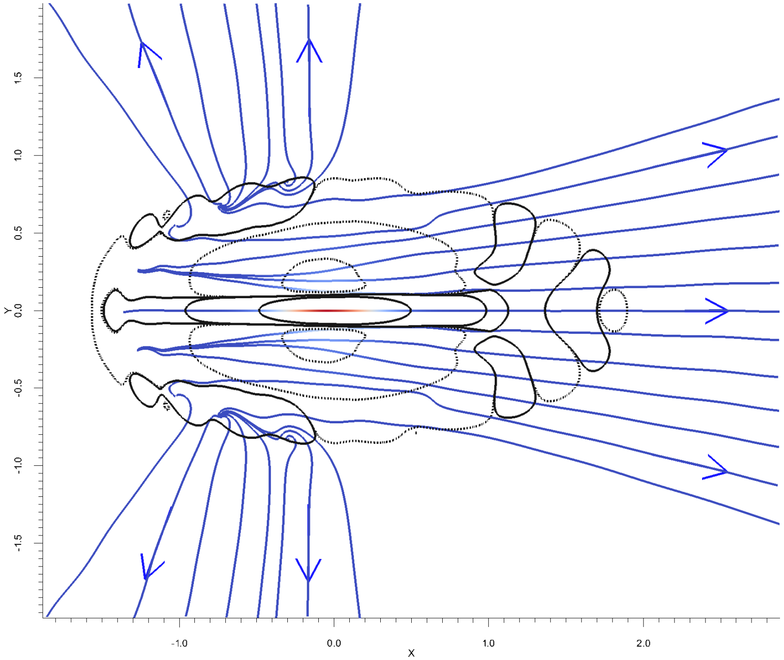
Trapped TFE:

- Generated on wavepacket axis,
- Transported by solenoidal momentum fluctuations,
- Attenuated in the sink.

Flux due to irrotational momentum fluctuations

$$(A = 0.01, \epsilon = 10)$$

$$\frac{\partial}{\partial x_j}\left(\overline{H'B'_j} + \overline{H'\frac{\partial \psi'}{\partial x_j}}\right) = \overline{\frac{\partial \psi'}{\partial x_i}(\vec{\Omega} \wedge \vec{u})'_i} - \overline{B'_i(\vec{\Omega} \wedge \vec{u})'_i} + \overline{E_{S_i}}.$$



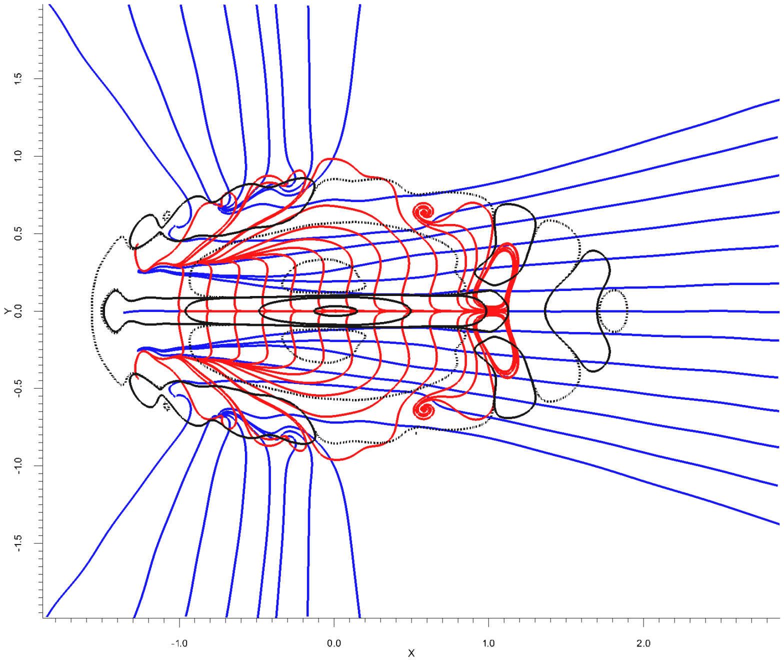
Propagating TFE:

- Generated at the core of the wavepacket
- Transported by irrotational momentum fluctuations
- Escapes in a downstream radiation lobe and two normal radiation lobes

Solenoidal and irrotational flux

$(A = 0.01, \epsilon = 10)$

$$\frac{\partial}{\partial x_j}\left(\overline{H'B'_j} + \overline{H'\frac{\partial \psi'}{\partial x_j}}\right) = \overline{\frac{\partial \psi'}{\partial x_i}(\vec{\Omega} \wedge \vec{u})'_i} - \overline{B'_i(\vec{\Omega} \wedge \vec{u})'_i} + \overline{E_{S_i}}.$$

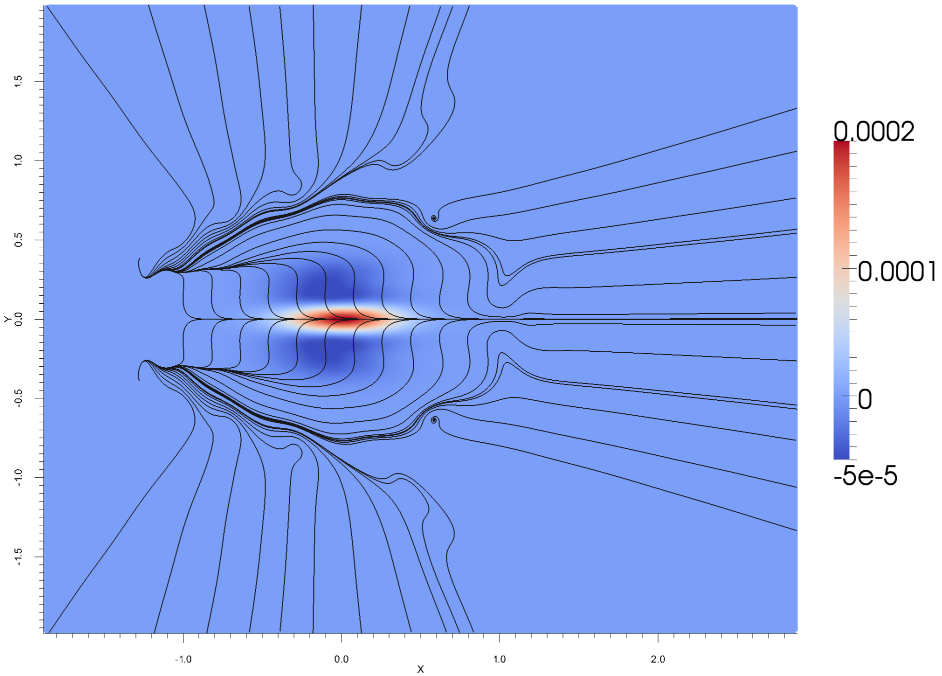


Doak's formulation enables separation and visualisation of these two different energy flux mechanisms.

Source decomposition

$$\overline{E_{S_i}} = \overline{\rho \mathbf{u} \cdot \mathbf{a}_f} = \overline{(\overline{B}_i + B_i' + \frac{\partial \psi'}{\partial x_i})a_i} + \overline{(\overline{B}_i + B_i' + \frac{\partial \psi'}{\partial x_i})\epsilon a_i^{\epsilon}}$$

$\overline{E_{S_i}}$



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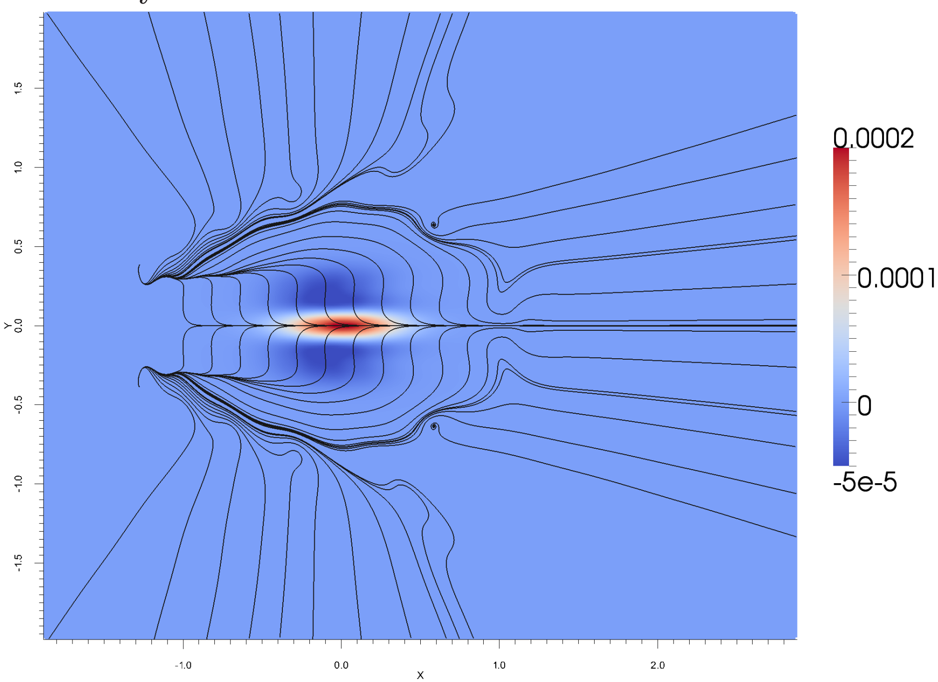

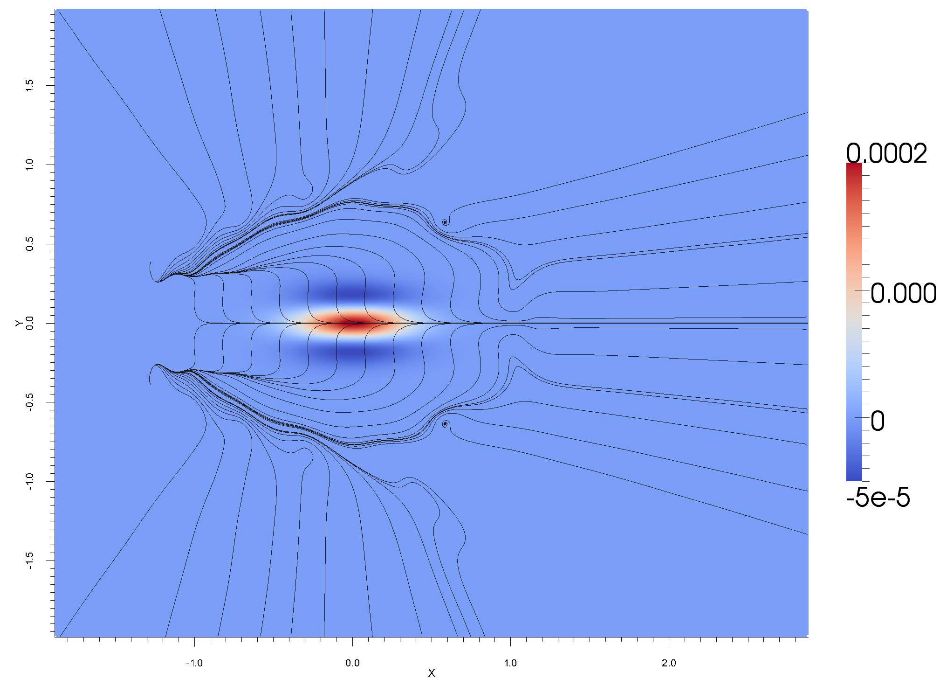

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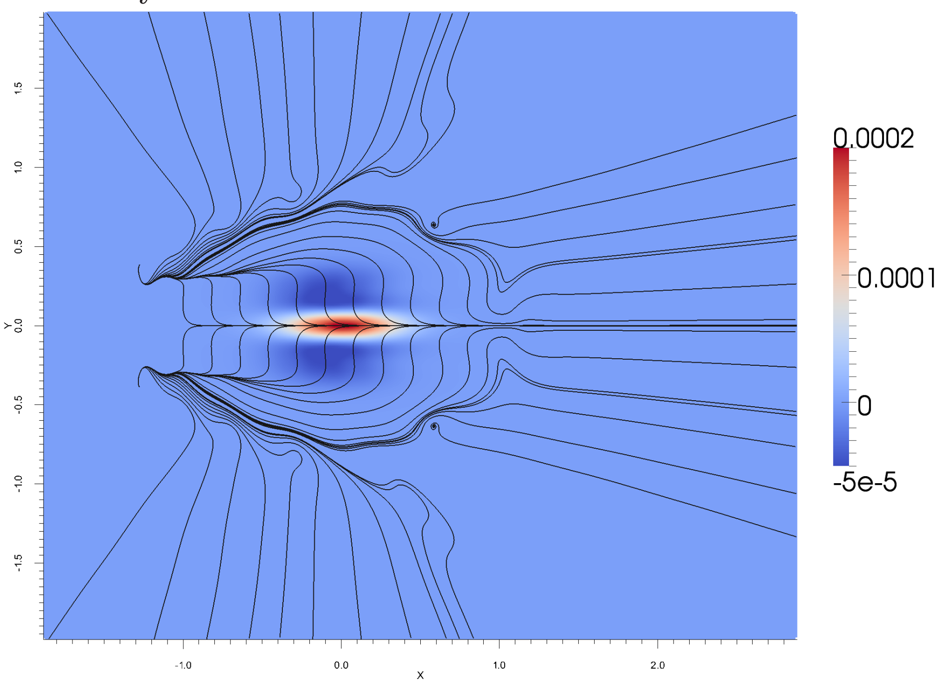

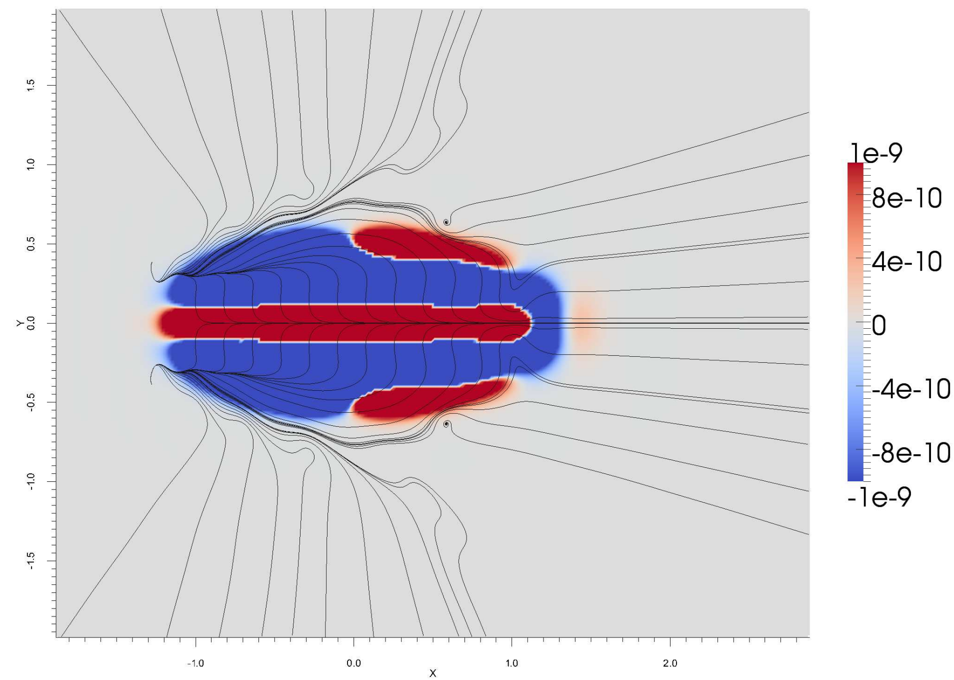

(b)

Source decomposition

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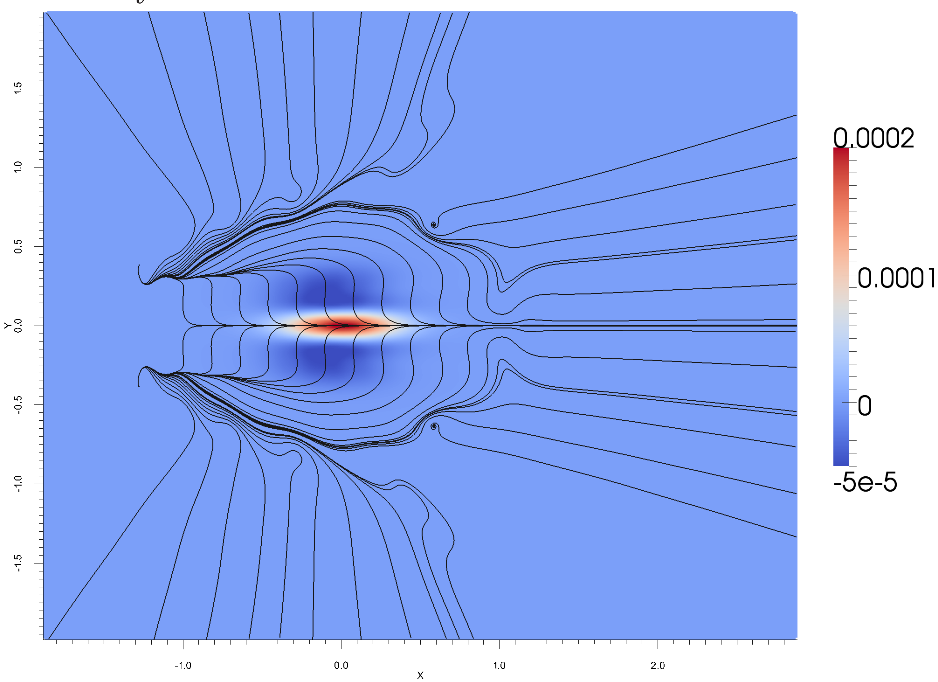
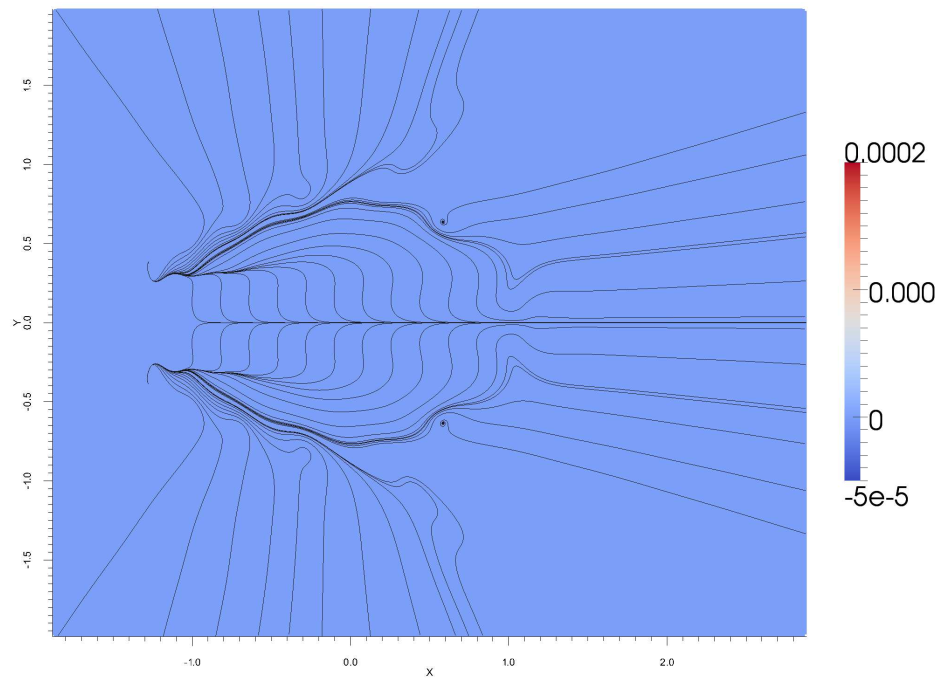
$\overline{E_{S_i}}$





Source decomposition

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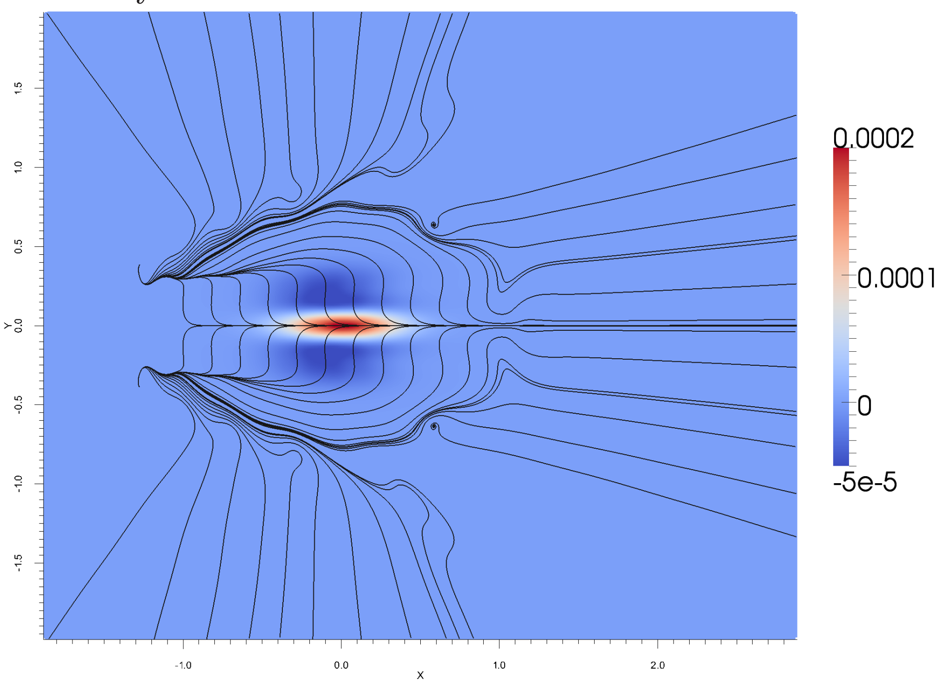
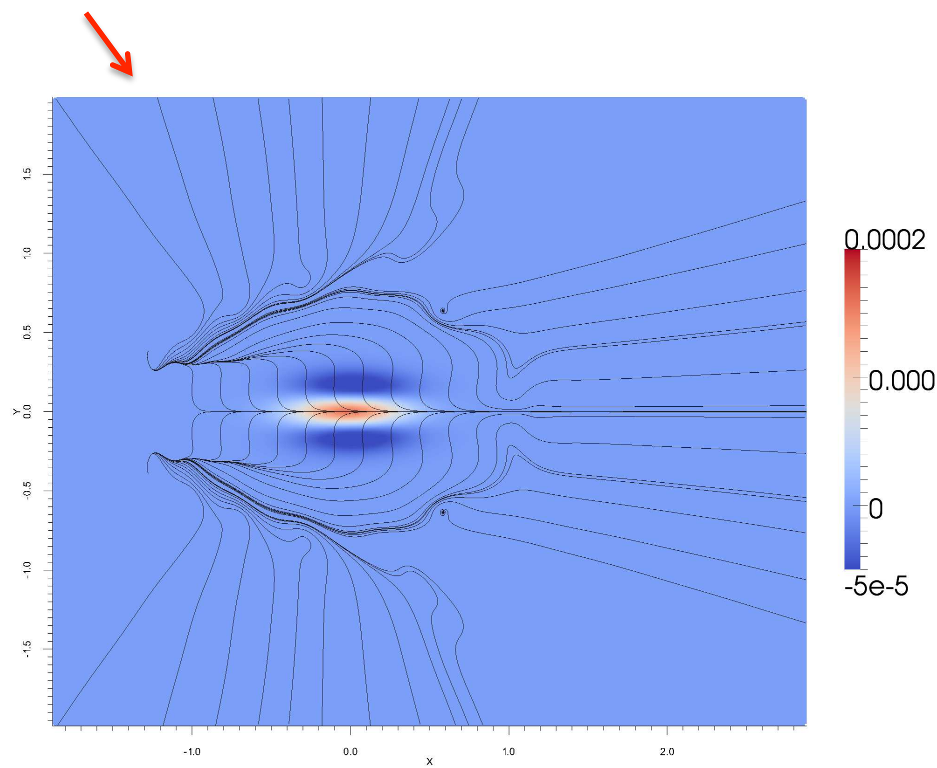
$\overline{E_{S_i}}$





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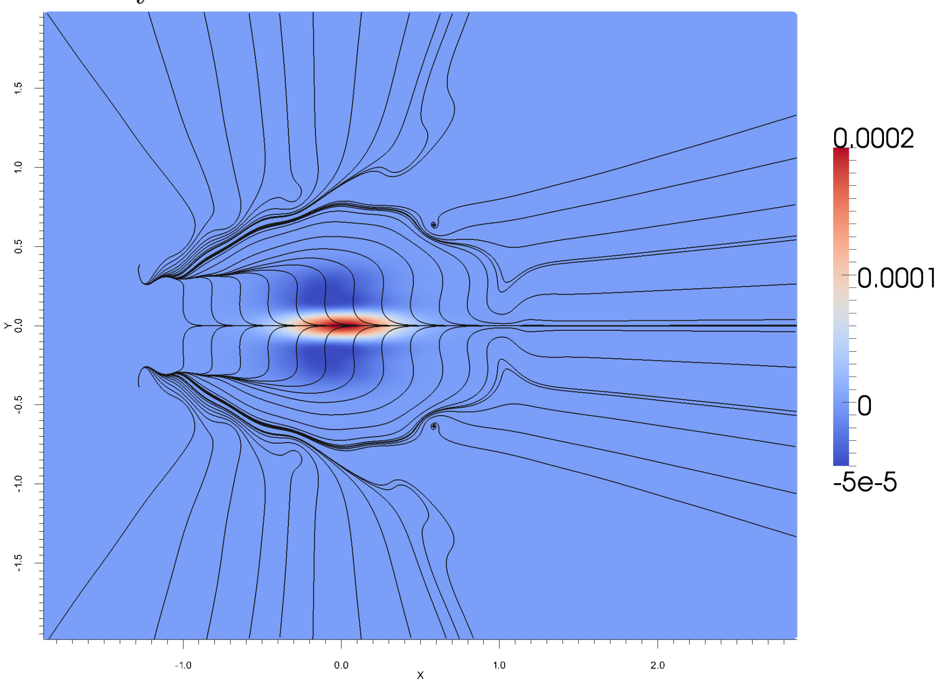
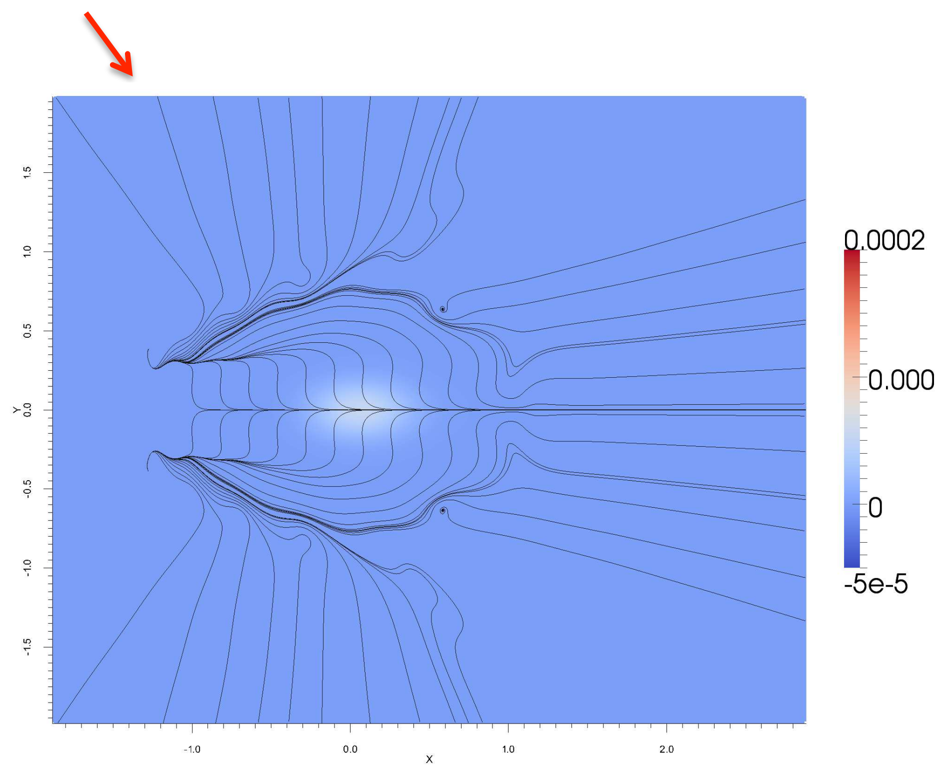
$\overline{E_{S_i}}$





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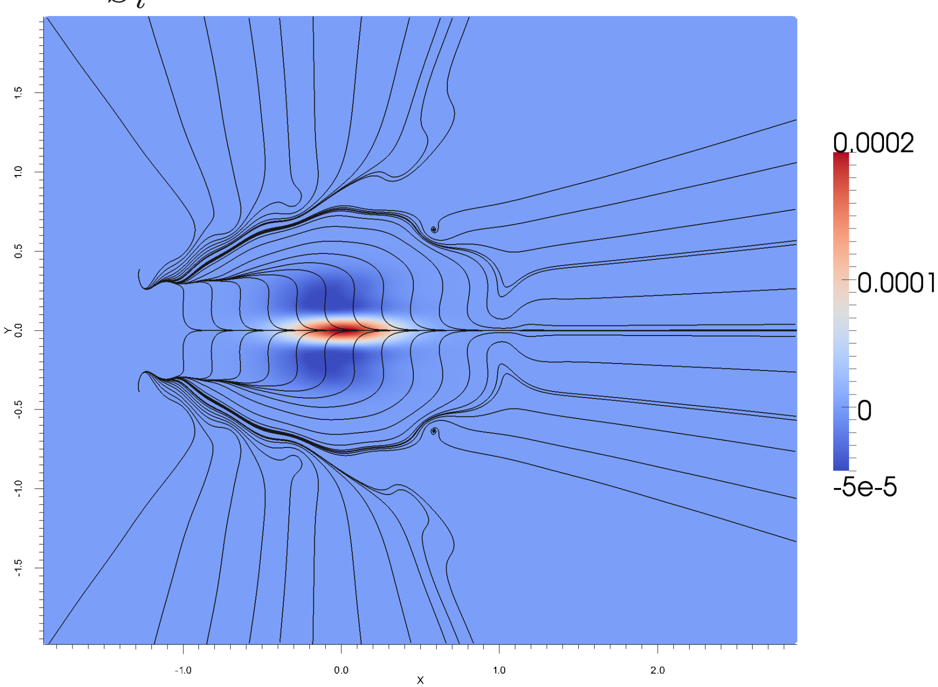
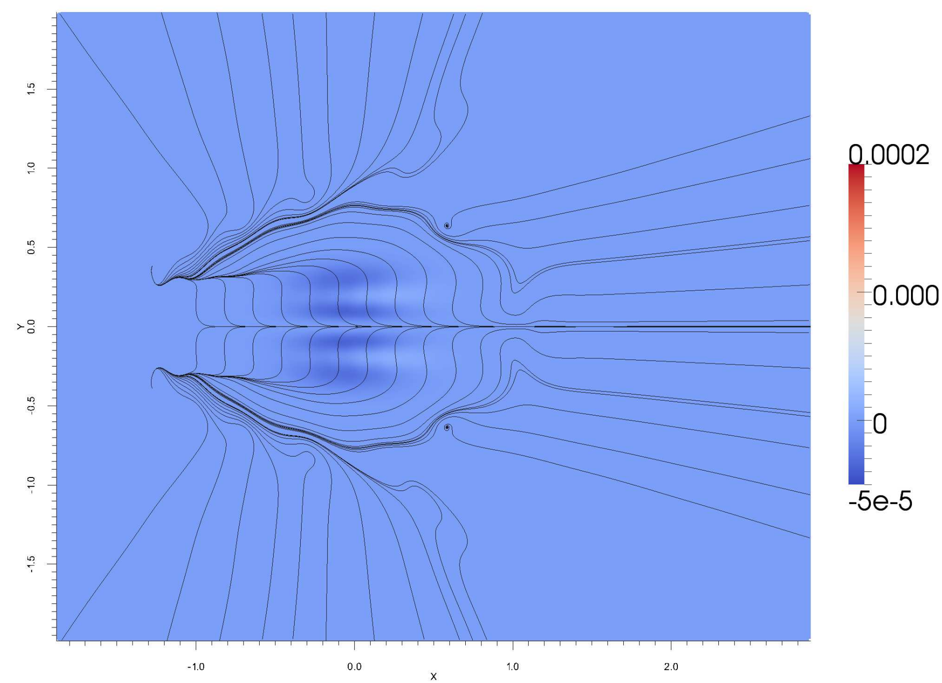
$\overline{E_{S_i}}$





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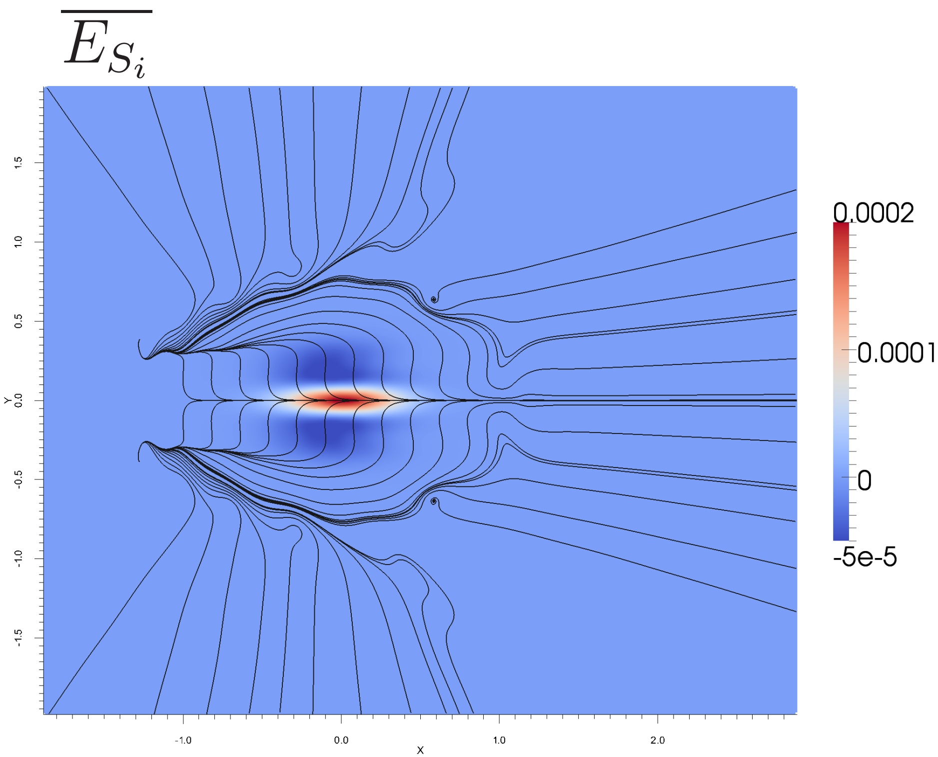

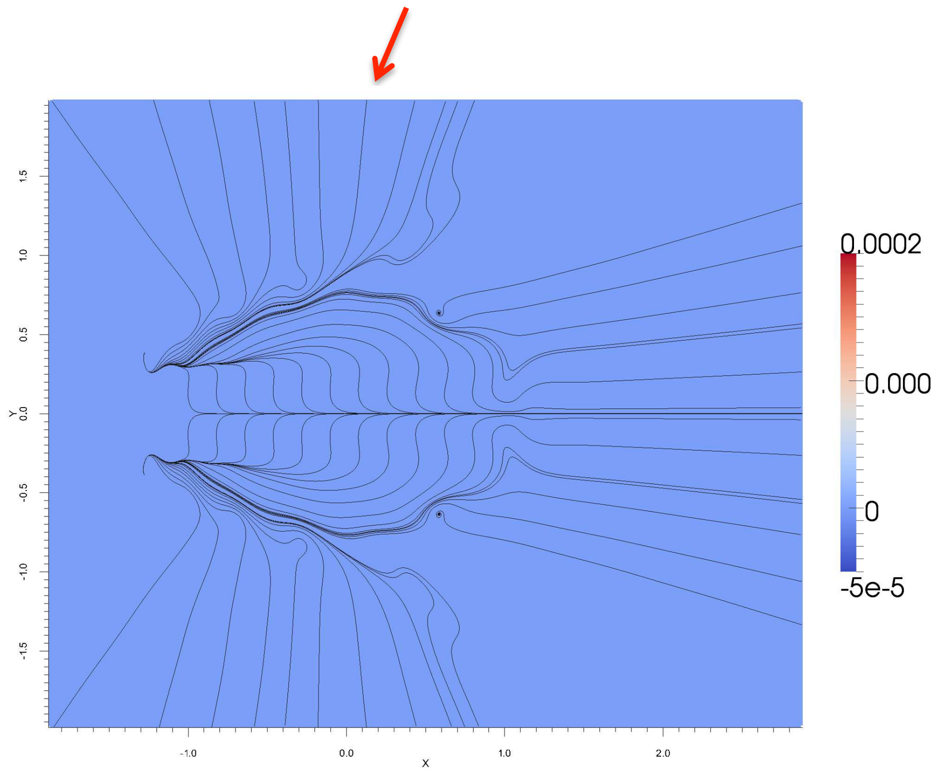

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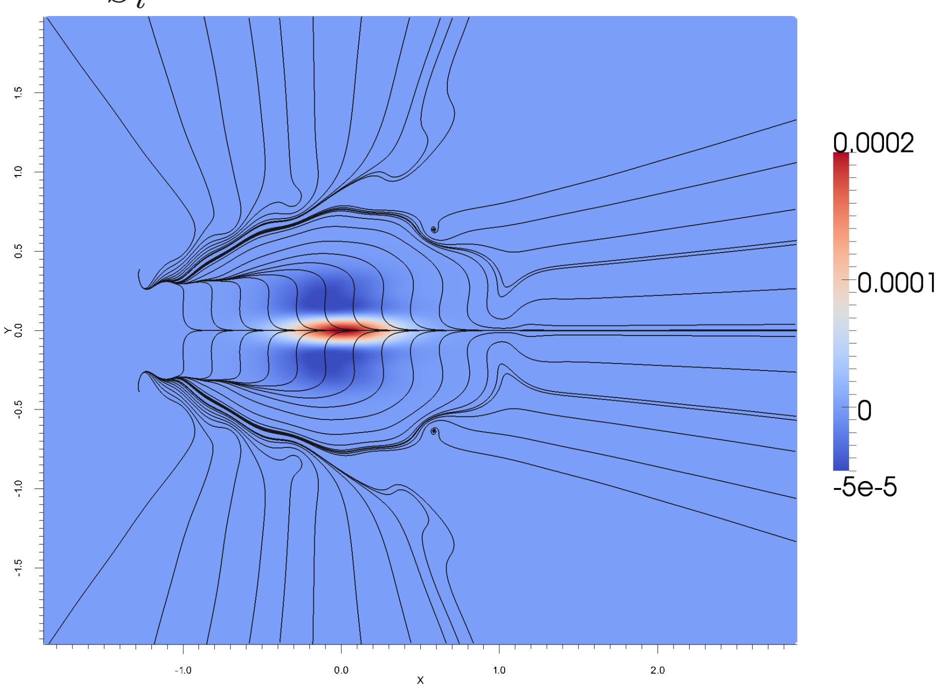
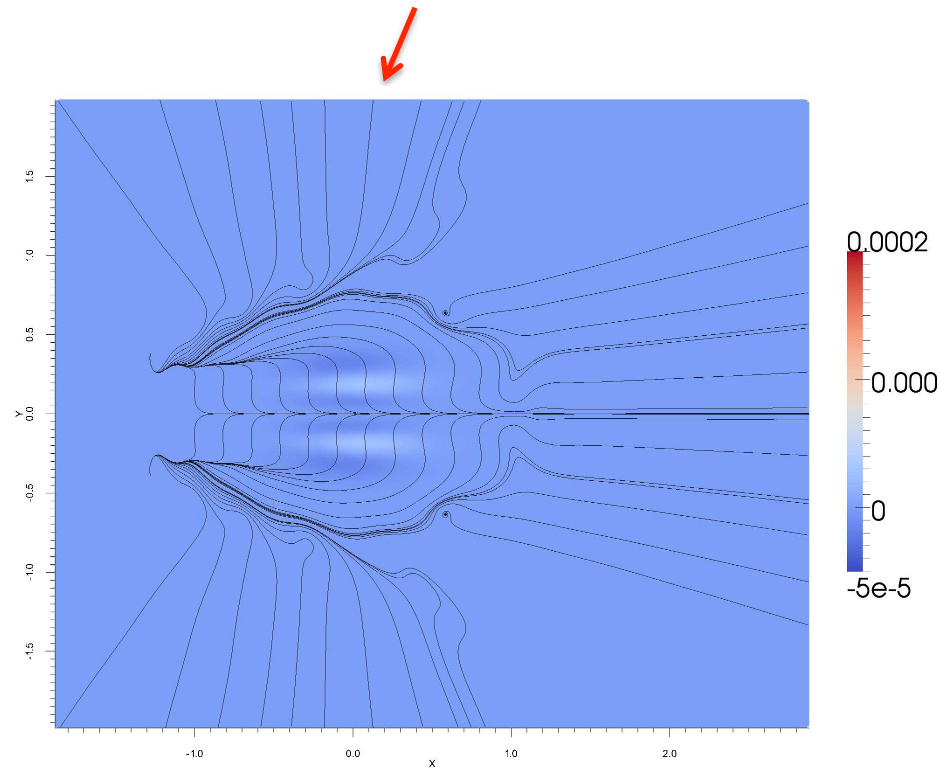
$\overline{E_{S_i}}$





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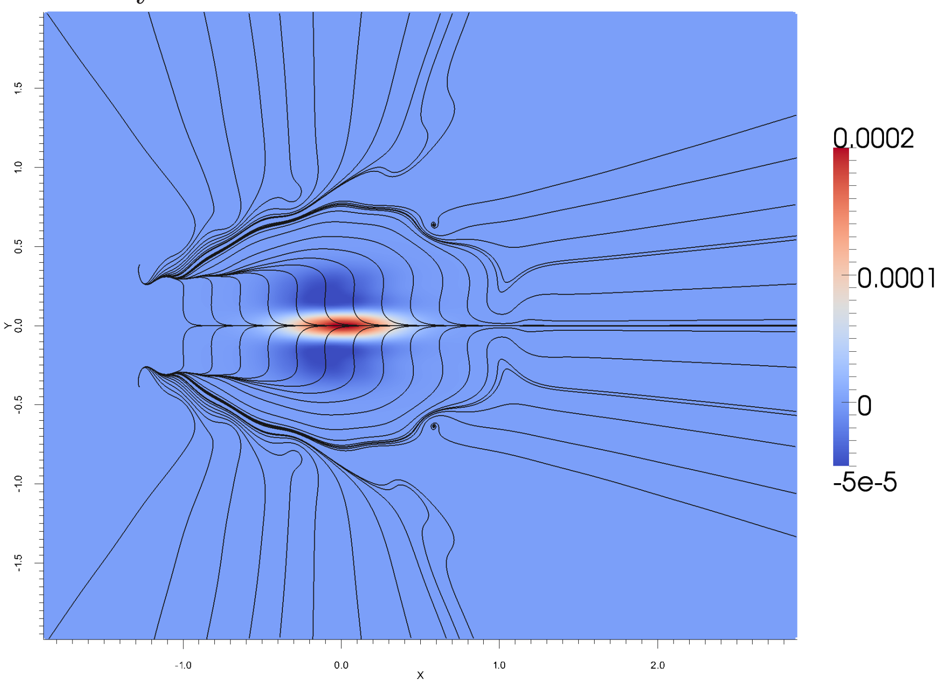
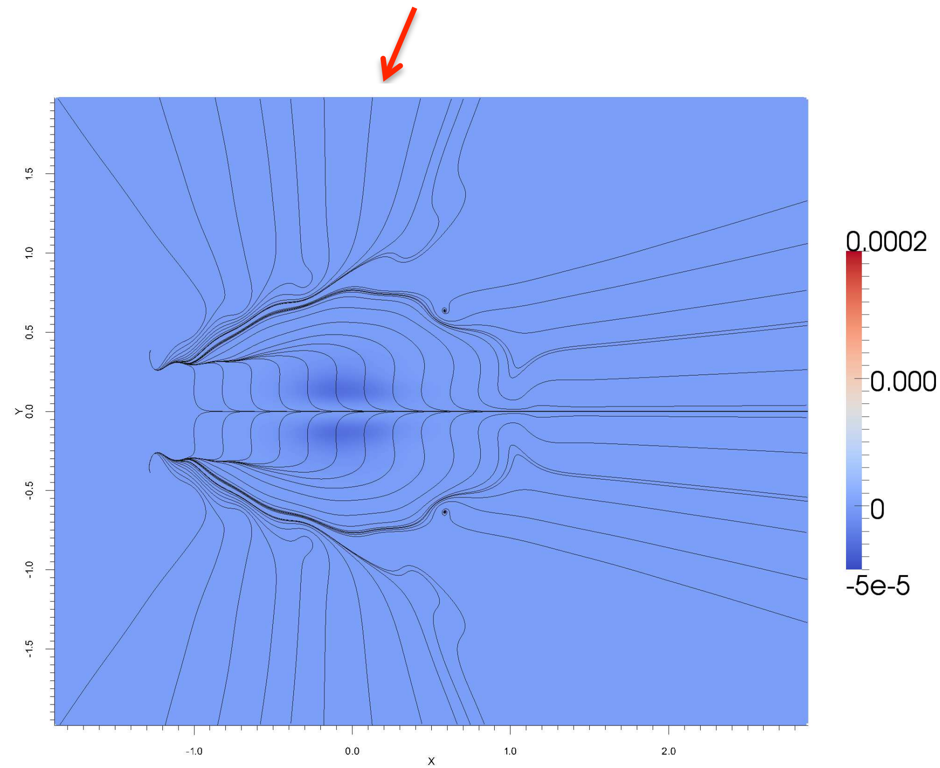
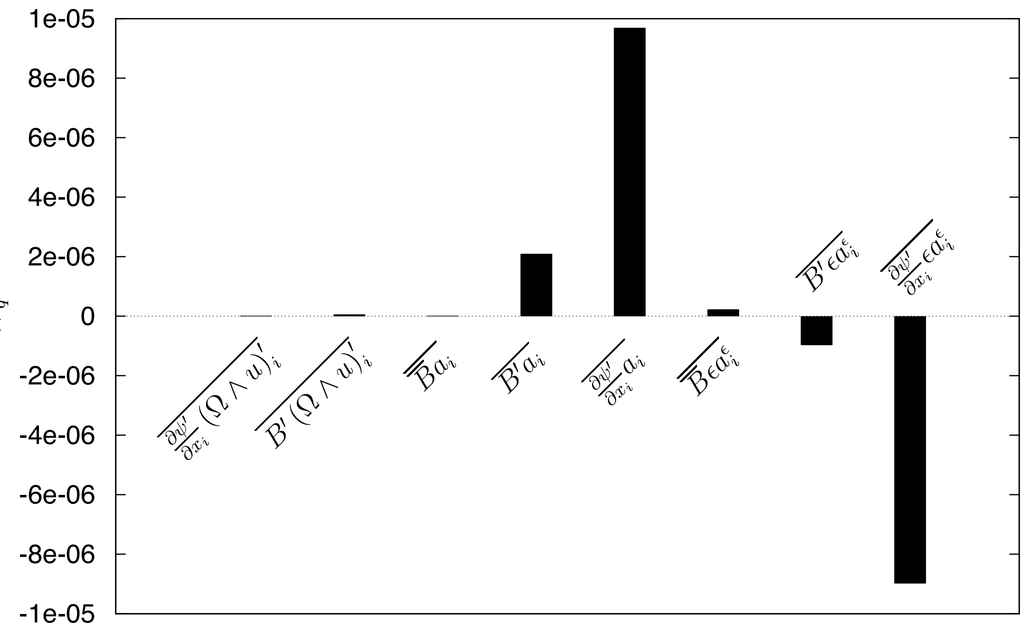
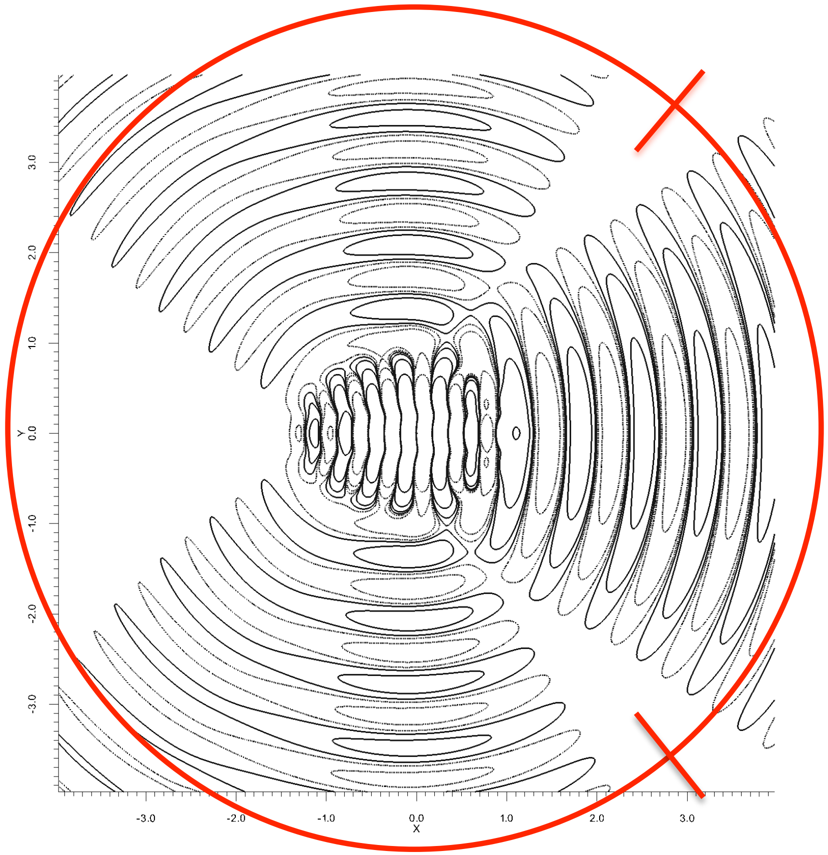
$\overline{E_{S_i}}$





Source contribution to propagating TFE





Conclusions

Doak's formalism allowed us to probe the inner workings of the sound generation mechanisms in the very simple flow considered,

The separation of *trapped* and *propagating* TFE,

Source mechanisms:

- Familiar downstream lobe appears to be due to alignement of wavepacket forcing with irrotational momentum fluctuations,

- Less familiar sidelobes appear to be due scattering of wavepacket motion by solenoidal momentum fluctuation

Would be useful to extend to:

- Systems with non-zero mean flow,
- Heated and multi-species flows, to study the entropy and other terms,
- Flow systems with stronger internal sources – compressible turbulence for instance…

Doak's momentum-potential theory of energy flux

Three simplifying properties for time-stationary flows:

1. Primary dependent vector field is expressed as linear superposition of mean solenoidal, fluctuating solenoidal and fluctuating irrotational components

$$\rho u_i = \overline{B}_i + B'_i - \frac{\partial \psi'}{\partial x_i}$$

2. Mass conservation reduces to linear Poisson equation

$$\frac{\partial^2 \psi'}{\partial x_i^2} = \frac{\partial \rho'}{\partial t}.$$

3. Scalar momentum potential has zero mean