Doak’s momentum potential theory of energy flux used to study a solenoidal wavepacket

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Abstract

Doak’s momentum potential theory of energy flux is explored using a model problem constructed to facilitate the introduction of solenoidal perturbations, in a controlled manner, to an otherwise well-understood sound generation problem: that of an irrotational, subsonically-convecting wavepacket. The solenoidal wavepacket has, in addition to a downstream radiation lobe similar to its irrotational counterpart, lower level radiation in the sideline direction. Helmholtz decomposition of the linear momentum and subsequent exploration of the various source and flux terms that participate, according to Doak’s fluctuation-energy corollary, in the generation and transport of total fluctuating enthalpy (TFE), reveal a rich inner structure involving work performed on, and extracted from, the fluid system by means of the irrotational and solenoidal components of the wavepacket. The response of the fluid involves the internal transport and attenuation of ‘trapped’ TFE, as well as the radiation of a small amount of ‘radiating’ TFE. The analysis shows how the downstream radiation is associated with a mechanism similar to that of the irrotational wavepacket, while the sideline radiation arises due to the scattering of irrotational wavepacket fluctuations by solenoidal momentum fluctuations. It is postulated that such a mechanism might play a role in the sideline radiation of subsonic jets.

1. Introduction

Phil Doak was fascinated by the question ‘How does turbulent fluid motion generate sound?’, and in particular by the dilemma posed by the fact that this question is really only meaningful if we know what is meant by ‘sound’ within the confines of the turbulent field. A sound wave is a...
Acoustic and other motions

\[ \mathcal{L}(u_s) = 0 \]
\[ \mathcal{L}(p) = 0 \]
\[ \mathcal{L}(\sigma) = 0 \]

Rayleigh (1877)
Chu & Kovaznay (1958)
Cantrell & Hart (1964)
Morfey (1971)
Doak (1974, 1989)
Pierce (1981)
Jenvey (1989)
Myers (1994)
Goldstein (2003, 2005)
Doak’s momentum-potential theory of energy flux

Linear momentum density: sum of solenoidal and irrotational components

\[ \rho v_i = B_i - \partial \psi / \partial x_i, \quad \partial B_i / \partial x_i = 0, \]

Mass conservation is then a linear equation:

\[ \partial \rho / \partial t - \partial^2 \psi / \partial x_i^2 = 0. \]

Assume time-stationarity in mass density fluctuations:

\[ \partial \rho' / \partial t - \partial^2 \psi' / \partial x_i^2 = 0, \quad \partial^2 \bar{\psi} / \partial x_i^2 = 0 \]

And so, for any time-stationary motion

\[ \rho v_i = \overline{\rho v_i} + (\rho v_i)' = \tilde{B}_i(x_k) + B'_i(x_k, t) - \partial \psi'(x_k, t) / \partial x_i \]

\[ \partial \tilde{B}_i / \partial x_i = \partial B_i / \partial x_i = 0. \]
Doak’s momentum-potential theory of energy flux

From the energy conservation equation:

\[
\frac{\partial}{\partial x_j} \left( H^j B_j + H^j B'_j - H' \frac{\partial \psi'}{\partial x_j} \right) = \overline{E_{S_i}}
\]

where \( H \) is the total enthalpy.

For fluctuating quantities only:

\[
\frac{\partial}{\partial x_j} \left( H' B'_j - H' \frac{\partial \psi'}{\partial x_j} \right) = \frac{\partial \psi'}{\partial x_i} (\tilde{\Omega} \land \vec{u})'_i - B'_i (\tilde{\Omega} \land \vec{u})'_i + \overline{E_{S_i}}.
\]

cf. Moyal (1952); Lighthill (1952); Chu & Kovaznay (1958).
A model problem: wavepacket forcing with vorticity enhancement

\[
\vec{u} = (u + \varepsilon u_\varepsilon, v + \varepsilon v_\varepsilon)
\]

\[
u = \frac{\partial \phi}{\partial x}; v = \frac{\partial \phi}{\partial y}
\]

\[
u_\varepsilon = -\frac{\partial \psi}{\partial y}; v_\varepsilon = \frac{\partial \psi}{\partial x},
\]

\[
\phi = A \exp(-\frac{(x - x_o)^2}{\lambda_x^2} - \frac{(y - y_o)^2}{\lambda_y^2}) \cos(k_x x - k_x U_c t),
\]

\[
\psi = y \phi,
\]
A model problem: wavepacket forcing with vorticity enhancement

\[ (A = 0.01, \epsilon = 10) \]

\[ \phi = A \exp\left(-\frac{(x-x_o)^2}{\lambda_x^2} - \frac{(y-y_o)^2}{\lambda_y^2}\right) \cos(k_x x - k_x U_c t), \]

\[ \psi = y\phi, \]
A model problem: flux and source terms

\[
\frac{\partial}{\partial x_j} \left( H' B'_j - H' \frac{\partial \psi'}{\partial x_j} \right) = \frac{\partial \psi'}{\partial x_i} (\Omega \wedge \vec{u})'_i - B'_i (\Omega \wedge \vec{u})'_i + E_{S_i}.
\]

\[
E_{S_i} = \rho u \cdot a_f = (B_i + B'_i + \frac{\partial \psi'}{\partial x_i}) a_i + \left(B_i + B'_i + \frac{\partial \psi'}{\partial x_i}\right) \epsilon a_i^\epsilon.
\]
A model problem: response of 2D Euler equations

\((A = 0.01, \epsilon = 0)\) \hspace{1cm} \((A = 0.01, \epsilon = 10)\)

\[
\overline{E_{S_i}} = \overline{\rho u \cdot a_f} = (\overline{B_i} + B'_i + \frac{\partial \psi'}{\partial x_i})a_i + (\overline{B_i} + B'_i + \frac{\partial \psi'}{\partial x_i})\epsilon a_i^\epsilon.
\]
Wavepacket without vorticity enhancement: source and flux terms

\[(A = 0.01, \epsilon = 0)\]

\[
\frac{\partial}{\partial x_j} \left( \frac{H'B_j'}{H'} - \frac{H'}{\partial x_j} \right) = \frac{\partial \psi'}{\partial x_i} (\Omega \wedge \mathbf{u})_i - B'(\Omega \wedge \mathbf{u})'_i + E_{S_i}.
\]

\[
E_{S_i} = (B_i + B'_i + \frac{\partial \psi'}{\partial x_i})a_i + (B_i + B'_i + \frac{\partial \psi'}{\partial x_i})\epsilon a_i^e.
\]
Wavepacket with vorticity enhancement: source and flux terms

\[(A = 0.01, \epsilon = 10)\]

\[
\frac{\partial}{\partial x_j}(\overline{H'B'_j} - H'\frac{\partial \psi'}{\partial x_j}) = \frac{\partial \psi'}{\partial x_i}(\vec{\Omega} \wedge \vec{u})'_i - B'_i(\vec{\Omega} \wedge \vec{u})'_i + \overline{E_{Si}}.
\]

**Source:** work performed on fluid by forcing.

**Sink:** work removed from fluid by forcing
Wavepacket with vorticity enhancement: source and flux terms

\( (A = 0.01, \epsilon = 10) \)

\[
\frac{\partial}{\partial x_j} \left( \overline{H' B'_j} - H' \frac{\partial \psi'}{\partial x_j} \right) = \frac{\partial \psi'}{\partial x_i} \left( \overline{\Omega \times \bar{u}} \right)_i - B'_i \left( \overline{\Omega \times \bar{u}} \right)_i + E_{S_i}.
\]

Source: work performed on fluid by forcing.

Sink: work removed from fluid by forcing
Flux due to solenoidal momentum fluctuations

\( (A = 0.01, \epsilon = 10) \)

\[
\frac{\partial}{\partial x_j} \left( \frac{H' B'_j}{H'} \frac{\partial \psi'}{\partial x_j} \right) = \frac{\partial \psi'}{\partial x_i} \left( \mathbf{\Omega} \wedge \mathbf{u}' \right)_i - \frac{B'_i}{x} \left( \mathbf{\Omega} \wedge \mathbf{u}' \right)_i + E_{S_i}.
\]

Trapped TFE:

- Generated on wavepacket axis,
- Transported by solenoidal momentum fluctuations,
- Attenuated in the sink.
Flux due to irrotational momentum fluctuations

\[ (A = 0.01, \epsilon = 10) \]

\[
\frac{\partial}{\partial x_j} \left( H' B_j' - H' \frac{\partial \psi'}{\partial x_j} \right) = \frac{\partial \psi'}{\partial x_i} (\vec{\Omega} \wedge \vec{u})'_i - B'_i (\vec{\Omega} \wedge \vec{u})'_i + \vec{E}_{S_i}.
\]

Propagating TFE:

- Generated at the core of the wavepacket
- Transported by irrotational momentum fluctuations
- Escapes in a downstream radiation lobe and two normal radiation lobes
Solenoidal and irrotational flux

\( (A = 0.01, \epsilon = 10) \)

\[
\frac{\partial}{\partial x_j} \left( H' B_j' + H' \frac{\partial \psi'}{\partial x_j} \right) = \frac{\partial \psi'}{\partial x_i} (\Omega \wedge \bar{u})'_i - B'_i (\Omega \wedge \bar{u})'_i + E_{S_i}.
\]

Doak’s formulation enables separation and visualisation of these two different energy flux mechanisms.
Source decomposition

\[
\overline{E_{S_i}} = \rho \mathbf{u} \cdot \mathbf{a}_f = \left( \overline{B_i} + B'_i + \frac{\partial \psi'}{\partial x_i} \right) a_i + \left( \overline{B_i} + B'_i + \frac{\partial \psi'}{\partial x_i} \right) \epsilon a^\xi,
\]

![Image](image-url)
\[ \overline{E_{S_i}} = \rho \mathbf{u} \cdot \mathbf{a}_f = (\overline{B_i} + B'_i + \frac{\partial \psi'}{\partial x_i})a_i + (\overline{B_i} + B'_i + \frac{\partial \psi'}{\partial x_i})\epsilon a^\xi. \]
Source decomposition

\[
\overline{E_{S_i}} = \rho \mathbf{u} \cdot \mathbf{a}_f = (\overline{B_i} + B'_i + \frac{\partial \psi'}{\partial x_i})a_i + (\overline{B_i} + B'_i + \frac{\partial \psi'}{\partial x_i})\epsilon a_i^\xi;
\]
Source decomposition

\[
\overline{E_{S_i}} = \overline{\rho u \cdot a_f} = (\overline{B_i} + B'_i + \frac{\partial \psi'}{\partial x_i})a_i + (\overline{B_i} + B'_i + \frac{\partial \psi'}{\partial x_i})\epsilon a^\xi:
\]

\[
\overline{E_{S_i}}
\]
Source decomposition

\[ \overline{E_{S_i}} = \rho u \cdot a_f = (\overline{B_i} - B'_i) \frac{\partial \psi'}{\partial x_i} a_i + (\overline{B_i} + B'_i + \frac{\partial \psi'}{\partial x_i}) \epsilon a_i^\xi; \]
Source decomposition

\[
\overline{E_{S_i}} = \bar{\rho} \mathbf{u} \cdot \mathbf{a}_f = (\overline{B_i} + B'_i + \frac{\partial \psi'}{\partial x_i}) a_i + (\overline{B_i} + B'_i + \frac{\partial \psi'}{\partial x_i}) \epsilon a^\epsilon_i;
\]

\[
\overline{E_{S_i}}
\]
Source decomposition

\[ \overline{E_{S_i}} = \rho \mathbf{u} \cdot \mathbf{a}_f = (\overline{B}_i + B'_i + \frac{\partial \psi'}{\partial x_i})a_i + (\overline{B}_i + B'_i + \frac{\partial \psi'}{\partial x_i})\epsilon a_i^\xi. \]
Source decomposition

\[ \overline{E_{S_i}} = \rho \mathbf{u} \cdot \mathbf{a}_f = (\overline{B_i} + B'_i + \frac{\partial \psi'}{\partial x_i})a_i + (\overline{B_i} + B'_i + \frac{\partial \psi'}{\partial x_i})\varepsilon \alpha_x; \]
Source decomposition

\[ \overline{E_{S_i}} = \rho u \cdot a_f = (\overline{B_i} + B'_i + \frac{\partial \psi'}{\partial x_i})a_i + (\overline{B_i} - B'_i + \frac{\partial \psi'}{\partial x_i})\epsilon a^\epsilon; \]

Figure 4: (a) Source decomposition

Figure 10 shows the structure of that part of the source associated with the external forcing function is constituted as follows:

\[ \overline{H_s} = \overline{p' \cdot u} = \left( \overline{B_i} + B'_i + \frac{\partial \psi'}{\partial x_i} \right) a_i + \left( \overline{B_i} - B'_i + \frac{\partial \psi'}{\partial x_i} \right) \epsilon a^\epsilon. \]
Source decomposition

\[ \overline{E_{S_i}} = \overline{\rho u \cdot a_f} = (\overline{B_i} + B'_i + \frac{\partial \psi'}{\partial x_i})a_i + (\overline{B_i} + B'_i + \frac{\partial \psi'}{\partial x_i}) \epsilon a_i^\epsilon. \]
Source contribution to propagating TFE

Figure 19: Contributions of each of the source terms to radiated sound.

This mechanism is similar to that comprised in the scattering of a plane wave by a cylindrical vortex filament, as described by [2].

Coherent structures in jets are frequently modelled using simplified, irrotational, wavepackets, and can successfully explain the mechanisms that underpin downstream radiation. They are less successful in explaining the sideline component. The study shows how the presence of vortical fluctuations—which are certainly carried by coherent structures in real jets—leads to an additional sound-production mechanism, associated with the scattering of irrotational fluctuations by the solenoidal component of the fluctuating momentum, and which radiates in the sideline direction. The study leads us to tentatively postulate that such a mechanism might play a role in the sideline radiation of turbulent jets.

7. Acknowledgements

This work is dedicated to the memory of Phil Doak, whose thinking motivated the study. The first two authors would also like to dedicate the paper to the memory of Pierre Comte, esteemed colleague and friend, who passed away last year. Both his fascination for, and intimate understanding of, compressible turbulence was an inspiration to us both.

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Conclusions

Doak’s formalism allowed us to probe the inner workings of the sound generation mechanisms in the very simple flow considered,

The separation of *trapped* and *propagating* TFE,

Source mechanisms:

Familiar downstream lobe appears to be due to alignment of wavepacket forcing with irrotational momentum fluctuations,

Less familiar sidelobes appear to be due scattering of wavepacket motion by solenoidal momentum fluctuation

Would be useful to extend to:

Systems with non-zero mean flow, Heated and multi-species flows, to study the entropy and other terms, Flow systems with stronger internal sources – compressible turbulence for instance...
Doak’s momentum-potential theory of energy flux

Three simplifying properties for time-stationary flows:

1. Primary dependent vector field is expressed as linear superposition of mean solenoidal, fluctuating solenoidal and fluctuating irrotational components

   \[ \rho u_i = \overline{B}_i + B'_i - \frac{\partial \psi'}{\partial x_i} \]

2. Mass conservation reduces to linear Poisson equation

   \[ \frac{\partial^2 \psi'}{\partial x_i^2} = \frac{\partial \rho'}{\partial t} \]

3. Scalar momentum potential has zero mean