Preferential concentration of impurities in turbulent suspensions

Jérémie Bec
CNRS, Observatoire de la Côte d’Azur, Nice

Small-scale Turbulence: Theory, Phenomenology and Applications, Cargèse, August 2007
In collaboration with

Luca Biferale (Rome)
Guido Boffetta (Turin)
Antonio Celani (Nice)
Massimo Cencini (Rome)
Raphaël Chétrite (Lyon)
Krzysztof Gawędzki (Lyon)
Rafaela Hillerbrand (Oxford)
Péter Horvai (Paris)
Alessandra Lanotte (Lecce)
Stefano Musacchio (Rehovot)
Federico Toschi (Rome)
Particle-laden flows

Finite-size and mass impurities advected by turbulent flow
Preferential concentration

Particles have **inertia** and do **not** follow exactly the fluid flow ⇒ they distribute **non-homogeneously**
Outline

Lecture 1

Introduction
Model, applications (rain, planets), questions

Small-scale clustering
Relation with dissipative dynamical systems, fractal attractors, etc.

Lecture 2

Inertial range clustering
Scale-dependence of inertia effects
Dispersed suspensions

- **Passive suspensions**: no feedback of the particles onto the fluid flow (e.g. very dilute suspensions)

- **Rigid spherical particles** that are assumed to
  
  (i) be much smaller than the smallest active scale of the flow (Kolmogorov scale in turbulence)
  
  (ii) have a very small Reynolds number

  ⇒ Surrounding flow = Stokes flow
  ⇒ Maxey & Riley (1983)

\[
m_p \ddot{X} = m_f \frac{Du}{Dt}(X, t) - 6\pi a \mu [\ddot{X} - u(X, t)] - \frac{m_f}{2} \left[ \dddot{X} - \frac{d}{dt} (u(X, t)) \right]
\]

+ Buoyancy
+ Faxén corrections

\[
- \frac{6\pi a^2 \mu}{\sqrt{\pi \nu}} \int_0^t \frac{ds}{\sqrt{t-s}} \frac{d}{ds} [\dddot{X} - u(X, s)].
\]
Very heavy inertial particles

• Impurities with mass density $\rho_p \gg \rho_f$

$$\ddot{X} = -\frac{1}{\tau} \left( \dot{X} - u(X, t) \right)$$

Prescribed velocity field (random or solution to Navier-Stokes)

• Viscous drag $\Rightarrow$ Response time (Stokes time): $\tau = \frac{2 \rho_p a^2}{9 \rho_f \nu}$

**Stokes number** $St = \frac{\tau}{\tau_\eta}$

$\tau_\eta$ turnover time associated to the Kolmogorov dissipative scale

$\tau_\eta = \frac{\eta}{\delta_\eta u} = \varepsilon^{-1/3} \eta^{2/3}$

• Minimal model: 2 parameters

$$\begin{cases}
St & \text{inertia} \\
Re & \text{turbulence intensity}
\end{cases}$$

$\Rightarrow$ allows for a systematic investigation
Clustering of heavy particles

• Important for estimating
  - particle interactions (collisions, chemical reactions, gravitation)
  - fluctuations in the concentration of a pollutant
  - possible feedback of the particles on the fluid

• Different mechanisms

  Dissipative dynamics  ⇒ **attractor**

  Ejection from *eddies* by centrifugal forces

**Philosophy**

- Describe with as much generality as possible clustering in turbulent flows
- Find models to disentangle these effects and understand their physics
Warm clouds
1 raindrop = $10^9$ droplets
Growth by continued condensation way
= too slow

Collisions
Polydisperse suspensions with a wide range of droplet sizes with different velocities
Larger, faster droplets overtake smaller ones and collide
Droplet growth by coalescence
Formation of the Solar system

**Protoplanetary disk** after the collapse of a nebula

(Ⅰ) Migration of dust toward the equatorial plane of the star

(Ⅱ) Accretion $\Rightarrow 10^9$ *planetesimals*
    from 100m to few km

(Ⅲ) Merger and growth
    $\Rightarrow$ *planetary embryos* $\Rightarrow$ planets

Problem = time scales?

From Bracco et al. (Phys. Fluids 1999)
Dissipative range clustering

- Stokes drag $\implies$ dissipative dynamics
  At large times, particle trajectories converge to a dynamically evolving attractor, which is in general multifractal.

- In other terms, the phase-space particle density $f(x, v, t)$ becomes singular in the asymptotic stationary regime.

- Relevant tools: borrowed from dissipative dynamical systems (Lyapunov exponents, fractal dimensions, etc.)
Attractor and mass distribution

- Dissipative system $\Rightarrow$ trajectories converge to a fractal attractor
  $f(x, v, t)$ density of particles is singular
  $\Rightarrow$ need for characterizing the particle distribution in terms of mass.

$$m_r(x, t) = \int_{|y|<r} f(x+y, t) \, dy \equiv \text{probability to have a particle in the ball of size } r$$

$x$ chosen as the position of a given particle
$$m_r(X(t), t) = m_r(t)$$
Fractal dimensions

- Scale-dependence of the moments of mass \( \langle m^p_r(t) \rangle \)
  \( \langle \cdot \rangle = \) average over all trajectories (i.e. w.r.t. the density \( f \))

- Examples: particles uniformly distributed
  - on a **curve** \( m_r \propto r \) \( \Rightarrow \) \( \langle m^p_r \rangle \sim r^p \)
  - on a **surface** \( m_r \propto r^2 \) \( \Rightarrow \) \( \langle m^p_r \rangle \sim r^{2p} \)
  - on a **fractal set** \( m_r \propto r^D \) \( \Rightarrow \) \( \langle m^p_r \rangle \sim r^{pD} \)

- Generically, mass is not uniformly distributed on the set but there are fluctuations \( \Rightarrow \) \( \langle m^p_r \rangle \sim r^{pD_{p+1}} \)

\( D_p = \) spectrum of dimension

(Grassberger, Hentschel-Procaccia 1983)

\( D_1 = \) information dimension \hspace{2cm} \( D_2 = \) correlation dimension
Tangent system

- **Linearization**
  \[
  \dot{X} = F(X, t) \Rightarrow \delta \dot{X} = D_X F(X(t), t) \delta X
  \]

- **1D:**
  \[
  \delta X(t) = \delta X(0) e^{\int_0^t D_X F(s) ds}
  \]
  \[
  \frac{1}{t} \ln \left| \frac{\delta X(t)}{\delta X(0)} \right| = \frac{1}{t} \int_0^t D_X F(s) ds \quad \text{Ergodicity} \quad t \to \infty
  \]

- **Generalization to multi-dimensional systems:**
  Linearized system:
  \[
  \delta X(t) = \mathcal{W}_{0,t} \delta X(0)
  \]
  \[
  \mathcal{W}^T_{0,t} \mathcal{W}_{0,t} \text{ symmetric positive matrix diagonalizes in } = Q_t^T \Lambda_t Q_t
  \]
  \[
  \Lambda_t = \text{diag}[e^{t \rho_1(t)}, \ldots, e^{t \rho_d(t)}] \quad \rho_1(t) > \cdots > \rho_d(t)
  \]
  \[
  \rho_i(t) = \text{stretching rates (or finite-time Lyapunov exponents)}
  \]
Lyapunov exponents

- **Oseledets** ergodic theorem: \( \rho_i(t) \to \lambda_i \) as \( t \to \infty \)
  \( \lambda_1, \ldots, \lambda_d = \) Lyapunov exponents
  \((\approx \text{law of large numbers})\)

- **Large deviations** of the stretching rates
  \[ p_t(\rho_1, \ldots, \rho_d) \propto e^{-t\mathcal{H}(\rho_1 - \lambda_1, \ldots, \rho_d - \lambda_d)} \]
  \( \mathcal{H} = \) rate function
  convex, attaining its minimum (equal to 0) at 0

\[ \mathcal{H} \]
\[ \lambda_i \]
\[ \rho_i \]
Lyapunov exponents and mass distribution

- $\lambda_1 =$ growth rate of an infinitesimal segment
  $\lambda_1 + \lambda_2 =$ growth rate on an infinitesimal surface
  $\lambda_1 + \lambda_2 + \lambda_3 =$ growth rate on an infinitesimal volume
  …
  $\lambda_1 + \cdots + \lambda_d =$ growth rate of phase-space volumes

- Chaotic dissipative systems:
  \[
  \begin{cases}
  \lambda_1 > 0 \\
  \lambda_1 + \cdots + \lambda_d < 0
  \end{cases}
  \]

Lyapunov dimension (Kaplan & Yorke, 1979)

\[
D_{KY} = J + \frac{\lambda_1 + \cdots + \lambda_J}{|\lambda_{J+1}|}
\]

Under some hypotheses: $D_{KY} = D_1$

(Ledrappier & Young, 1988)
Generalization to the dimension spectrum

Mass conservation:  $m_{r_1,r_2} = m_{r_1e^{-T\rho_1},r_2e^{-T\rho_2}}$

“Markovianity”  $\langle m_{r_1,r_2}^p \rangle = \langle \langle m_{r_1e^{-T\rho_1},r_2e^{-T\rho_2}}^p \rangle \rangle_{-\infty,t-T}[t-T,t]$

$\sim \langle r_1^p e^{-pT\rho_1} r_2^p (D_{p+1}-1) e^{-p(D_{p+1}-1)T\rho_2} \rangle_{[t-T,t]}$

$\Rightarrow \langle e^{-pT[\rho_1+(D_{p+1}-1)\rho_2]} \rangle \sim \text{const}$

Large deviations:  $T$ large  $\Rightarrow  p_T(\rho_1, \rho_2) \propto e^{-T\mathcal{H}(\rho_1-\lambda_1,\rho_2-\lambda_2)}$

$\int e^{-T[p\rho_1+p(D_{p+1}-1)\rho_2+\mathcal{H}(\rho_1-\lambda_1,\rho_2-\lambda_2)]} d\rho_1 d\rho_2 \sim \text{const}$

Saddle point:  $\min_{\rho_1, \rho_2} [p\rho_1 + p(D_{p+1}-1)\rho_2 + \mathcal{H}(\rho_1-\lambda_1,\rho_2-\lambda_2)] = 0$

(JB, Gawedzki & Horvai, 2004)
Lyapunov dimension

Asymptotics: \[ \text{St} \ll 1 \quad \mathcal{D}_{KY} \approx d - C \text{St}^2 \]

\[ \text{St} \gg 1 \quad \mathcal{D}_{KY} \to 2d \]

\[ R_\lambda = 110 \]

Falkovich et al. (2001)
Lyapunov dimension

Threshold in Stokes number for the presence of fractal clusters in physical space  (JB 2003)
Lyapunov dimension

$R_\lambda = 110$

Stokes Nu

$St < St_{cr}$
Lyapunov dimension

$R_\lambda = 110$

$St > St_{cr}$

Lyapunov dimension $d_{kY}$
Correlation dimension

- Estimated from \( \text{Prob}(|\mathbf{R}| < r) \propto r^{D_2} \)

Use of a tree algorithm to measure dimensions at scales \( \ll \eta \)

DNS (JB, Biferale, Cencini, Lanotte, Musacchio & Toschi, 2007)
Correlation dimension

![Correlation dimension graph](image)

- For $R_\lambda = 65$
- For $R_\lambda = 105$
- For $R_\lambda = 185$
Multifractal distribution

$D_p \neq D_q$

$\downarrow$

Intermittency in the mass distribution
Kraichnan flow

- Gaussian carrier flow with no time correlation
  
  Incompressible, homogeneous, isotropic

\[
\langle u_i(x, t) u_j(x', t') \rangle = [2D_0 \delta_{ij} - B_{ij}(x - x')] \delta(t - t')
\]

\[
B_{ij}(r) \approx D_1 \left[ (d + 1) r^2 \delta_{ij} - 2 r_i r_j \right]
\]

- No structure \( \Rightarrow \) effect of dissipative dynamics isolated

(JB, Cencini & Hillerbrand, 2007)
Reduction of the dynamics

\[ \begin{align*}
\frac{dX}{ds} &= -[X + X^2 - Y^2] \, ds + \sqrt{2\text{St}} \, dB_1 \\
\frac{dY}{ds} &= -[Y + 2XY] \, ds + \sqrt{6\text{St}} \, dB_2 \\
\frac{dR}{ds} &= XR \, ds
\end{align*} \]

with \( \text{St} = D_1 \tau \)

- Lyapunov exponent \( \lambda_1 = \langle X \rangle / \tau \)

Expansion in powers of the Stokes number

= diverging series \( \Rightarrow \) Borel resummation

Duncan, Mehlig, Östlund & Wilkinson (2005)
"Solvable" cases

• **One dimension** (Wilkinson & Mehlig 2004, Derevyanko et al. 2006)

Potential

\[ U(X) = \frac{X^2}{2} + \frac{X^3}{3} \]

Constant flux solution

\[ p_t(\rho_1) \propto e^{-tD_1St^{-2/3}h(St^{2/3}\rho_1/D_1)} \]

Lyapunov exponent:

\[ \lambda_1 = \frac{D_1}{2} \left[ -1 + \frac{St}{\alpha} \frac{Ai'(\alpha^2/St^2)}{Ai(\alpha^2/St^2)} \right] \]

• **Large-Stokes asymptotics**

( Horvai nlin.CD/0511023)

\[ \lambda_1 \propto D_1St^{-2/3} \]

+ same for stretching rate

\[ \rho_1(t) = \frac{1}{t} \ln[|R(t)|/|R(0)|] \]

\[ p_t(\rho_1) \propto e^{-tD_1St^{-2/3}h(St^{2/3}\rho_1/D_1)} \]

(JB, Cencini & Hillerbrand, 2007)
Lyapunov exponent in DNS

$R_\lambda \approx 185$

JB, Biferale, Boffetta, Cencini, Musacchio & Toschi (2006)

Same qualitative picture

Increase of chaoticity not reproduced in $\delta$-correlated flows
Correlation with the local flow structure

- Depending whether the eigenvalues of the strain matrix are real or complex conjugate: different local structure

![Diagram showing rotating and strain-dominated regions](image-url)
Inertial range clustering

Slice width
$\approx 2.5\eta$

Red = rotating regions

$\approx 30\eta$

$St = 1.34$
Coarse-grained mass

- $\rho_r = m_r / r^3$ = density averaged in a box of size $r$

$St$ increases or decreases

Algebraic tail

Tail faster than exponential
Small Stokes / Large box scaling

- The two limits $\tau_s \to 0$ and $r \to \infty$ are equivalent
- Naïve idea: **Local Stokes number**
  \[
  St(r) = \frac{\tau_s}{\varepsilon^{1/3} r^{2/3}}
  \]
  works in random self-similar flows
- Actually, scaling determined by the **increments of pressure**:
  Small inertia: Maxey’s approximation
  \[
  \dot{X} - u(X, t) = \tau \ddot{X} \approx a(X, t)
  \]
  $\Rightarrow$ synthetic compressible flow:
  \[
  \dot{X} \approx v(X, t) \quad v = u - \tau_s (\partial_t u + u \cdot \nabla u)
  \]
**Small Stokes / Large box scaling**

- Relevant time scale for the time evolution of a blob of particles

\[
\Gamma = \frac{1}{r^3} \int_{B_r} \nabla \cdot \mathbf{v} \, d^3x \sim -\frac{\tau_s}{r} \Delta_r \nabla p
\]

- Dimensional analysis: \(\Delta_r \nabla p \sim \varepsilon^{2/3} r^{-1/3}\)

Observed: scaling dominated by sweeping

\[
\Delta_r \nabla p \sim U \varepsilon^{1/3} r^{-2/3}
\]

so that

\[
\Gamma \propto \tau_s r^{-5/3}
\]
Small Stokes / Large box scaling

- The density distribution depends only on $\Gamma \propto \tau_s r^{-5/3}$.
Modeling mass dynamics

(JB & R. Chétrite, 2007)

• **Motivation**: understand how universal is the shape of the mass distribution observed

• Flow divided in cells. With a probability $p$ the cells are rotating and eject particles to their non-rotating neighbors

• Each cell contains a continuous mass $m_j$ of particles
Motivations for such a model

- Two dimensions: approximation = piecewise linear strain
  Particle dynamics in a constant vorticity $\omega$
  \[
  \frac{d^2 X}{dt^2} = -\frac{1}{\tau} \frac{dX}{dt} + \frac{\omega}{\tau} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X
  \]

- The distance from the center of the cell increases exponentially with rate
  \[
  \mu = \frac{-1 + \frac{1}{2} \sqrt{2\sqrt{1 + 16\tau^2\omega^2} + 2}}{2\tau}
  \]

- The mass in the cell of size decreases exponentially in time
  \[
  m(T) = m(0) (1 - \gamma) = m(0) \exp\left[ -\frac{T}{\tau} \left( -1 + \frac{1}{2} \sqrt{2\sqrt{1 + 16\tau^2\omega^2} + 2} \right) \right]
  \]
Ejection rate vs. Stokes number

\[ \gamma = 1 - \exp \left[ -\frac{K u}{St} \left( -1 + \frac{1}{2} \sqrt{2} \sqrt{1 + 16St^2} + 2 \right) \right] \]

\[ St = \tau \omega \]

Kubo \[ Ku = T \omega \]
One-dimensional version of the model

Steady state given by stationary solutions to the Markov equation

\[
m_j(n+1) = \begin{cases} 
  m_j(n) - \frac{\gamma}{2} \left[2 - \Omega_{j-1} - \Omega_{j+1}\right] m_j(n) & \text{if } \Omega_j = 1 \\
  m_j(n) + \frac{\gamma}{2} \left[\Omega_{j-1} m_{j-1}(n) + \Omega_{j+1} m_{j+1}(n)\right] & \text{if } \Omega_j = 0
\end{cases}
\]

\[
p_1(m) = \left[p^3 + (1-p)^3\right] p_1(m) + \frac{2p^2(1-p)}{1-\gamma/2} p_1 \left(\frac{m}{1-\gamma/2}\right) + \frac{p(1-p)^2}{1-\gamma} p_1 \left(\frac{m}{1-\gamma}\right) + 2p(1-p)^2 \int_0^{2m/\gamma} dm' p_2 \left(m', m - \frac{\gamma}{2} m'\right) + p^2(1-p) \int_0^{2m/\gamma} dm' \int_0^{2m/\gamma-m'} dm'' p_3 \left(m', m - \frac{\gamma}{2}(m' + m''), m''\right)
\]
One-cell mass distribution

PDF of $m_j$ very similar to that obtained in DNS (same tails)
Left tail

Algebraic behavior $p(m) \propto m^{\alpha(\gamma)}$ when $m \ll 1$

Mass-ejecting realization

\[ \begin{align*}
M \text{ times} & \quad \text{Final mass } m_j \approx (1 - \gamma)^N (1 - \gamma/2)^M \\
N \text{ times} & \quad \text{Initial mass } m_j \approx 1
\end{align*} \]

\[ \text{Prob} = \left[ p(1 - p)^2 \right]^N \left[ p(1 - p) \right]^M \]

Requirements that mass $m_j = m$ and that $\text{Prob}$ is maximal

$\Rightarrow$ Leading behavior at small masses
Left tail

$\alpha(\gamma, p)$ solution of

$$\frac{2p}{(1 - \gamma/2)^{\alpha+1}} + \frac{(1 - p)}{(1 - \gamma)^{\alpha+1}} = 3$$
\[ m = \frac{1 - [1 - (1 - \gamma/2)^N]^M}{(1 - \gamma/2)^N} \]

\[ M = M(m, N) = \frac{\log [1 - m(1 - \gamma/2)^N]}{\log [1 - (1 - \gamma/2)^N]} \]

Probability of such configurations

\[ \mathcal{P} = \left[ p^2(1 - p) \right]^{NM} = \exp \left[ \log(p^2(1 - p)) N M(m, N) \right] \]

Dominant contribution given by choosing \( N \) such that \( \mathcal{P} \) is maximal

\[ N^* \propto \log m \quad \Rightarrow \quad p(m) \propto \exp(-C m \log m) \]
Right tail
Coarse graining

\[ \tilde{m}_L = \frac{\ell}{L} \sum_{j=-K}^{K} m_j \]

Right tail
\[ p(\tilde{m}_L) \propto \exp(-C\tilde{m}_L \log \tilde{m}_L) \]

Left tail
\[ p(\tilde{m}_L) \propto \tilde{m}_L^{\alpha_L(\gamma)} \]

In both limits \( \gamma \to 0 \) and \( L \to \infty \) we recover uniformity.
Is there a rescaling in this limit?

\( \Rightarrow \) Prediction for the exponent of the left tail when \( L \gg \ell \)
\[ \alpha_L \approx \frac{1}{2} \frac{L}{\log(1 - \gamma/2)} \log \left[ p^4(1 - p)^2 \right] \]

\( \Rightarrow \) Picture repeated for
\[ \frac{L}{\log(1 - \gamma/2)} = \text{const} \]
Summary

Different mechanisms for clustering:

• Dissipative dynamics in the viscous range
  scale invariance
  relevant fluid time scale = Kolmogorov time
  insensitivity to flow intermittency
  use of tools borrowed from dynamical systems, model flows

• Ejection from vortices in the inertial range
  scale invariance broken
  relevant fluid time scale given by acceleration
  particle mass distribution has a universal shape
  need for better quantifying correlations with the flow structures and with the acceleration field
Open questions

**Collision / reaction rates** (important for applications)
  requires to quantify not only the distribution but also the velocity difference between particles

**Correlation with the flow structure**
  Inertial-range distribution of acceleration (pressure gradient) plays a crucial rôle

**Quantify inertial biases in particle tracking experiments**
  for e.g. acceleration, Lagrangian structure functions, etc.