Strongly anisotropic turbulence, structure and cascade
Stratified and rotating turbulence, mainly

***Summer School,* Cargèse**

Claude Cambon (with our team “ondes et turbulence”)

Laboratoire de Mécanique des Fluides et d’Acoustique
CNRS – ECL – UCB Lyon I

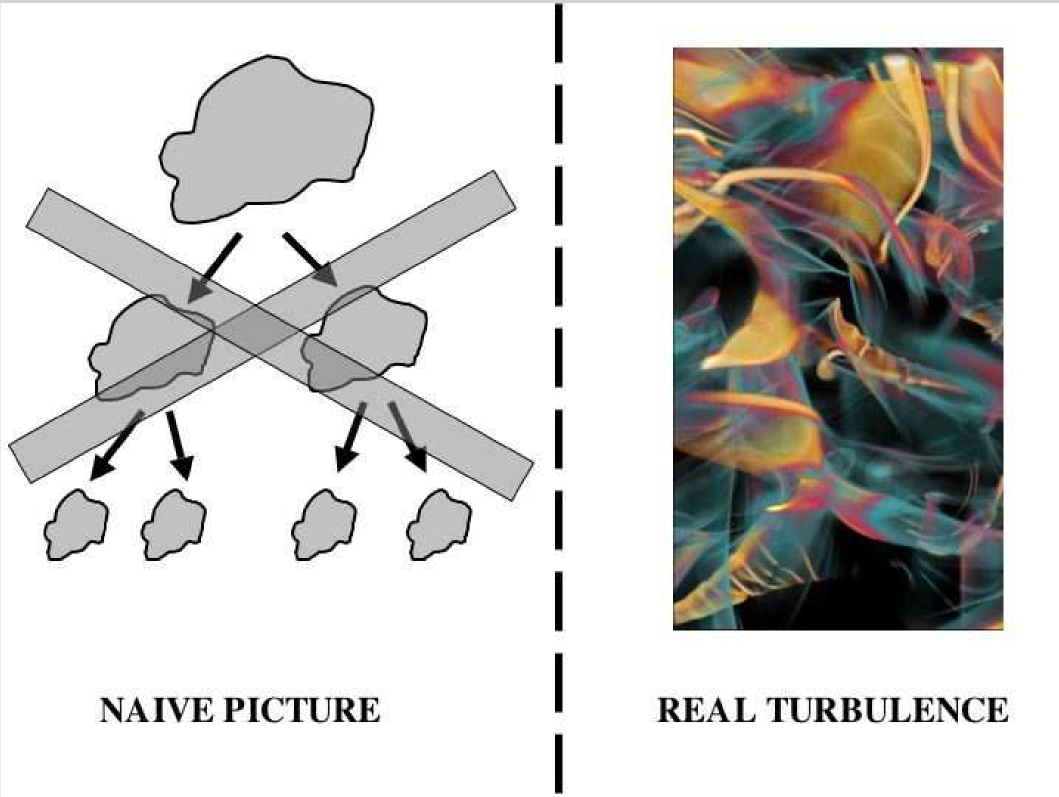
August 15 2007

CTRL-L switch

Collaboration and documents

- Team : Fabien S. Godeferd, Julian F. Scott, Lukas Liechtenstein, Alex Delache, Guillaume Simon, Benjamin Favier
- Future or in progress : Wouter Bos, (Luminita Danaila, Jan-Bert Flor, ANISO ?) Jose Redondo, Peter A. Davidson ...
- Documents (in addition to, or instead of, published articles)
  -) http://www.lmfa.ec-lyon.fr/Fabien.Godeferd/perso/ (Summer School in Barcelone)
  -) http://www.lmfa.ec-lyon.fr/Henri.Benard/news/ (Sagaut/Cambon book)

General considerations : what is the cascade ?



Refreshing our basic concept of cascade : not at all a dichotomic splitting, as in Frisch's book, but instead a complex effect of stretching, folding, engulfment, enrollment of velocity gradients, plus realistic vortex breakdown, reconnection, pairing ...

Quantifying the cascade

- ‘Modern’ approach : Kolmogorov $4/5$ law

$$\langle(\delta u_L)^3\rangle = -\frac{4}{5}\varepsilon r + 6\nu\frac{\partial}{\partial r}\langle(\delta u_L)^2\rangle$$

$R_{LLL}(r) = \langle(\delta u_L)^3\rangle$ is the key (third-order two-point) quantity

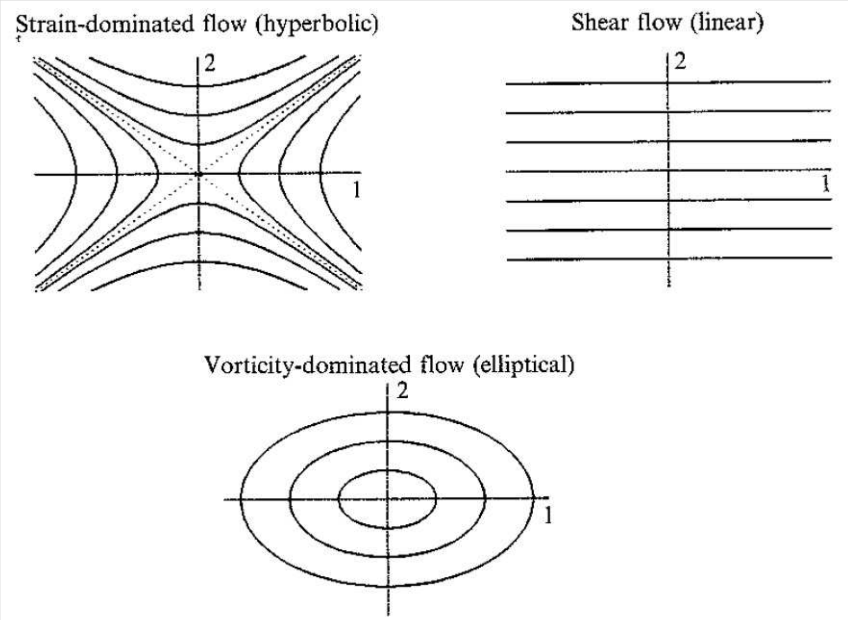
- ‘Old fashion’ approach : Lin equation

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) E = T$$

$T(k)$ can be related to the exact dynamics of *triads*, third-order three-points

Dependency on energy injection

- Realistic way: instabilities at large scale, ANISOTROPIC and inhomogeneous process,
- alteration of cascade by phase-mixing of waves, coexistence of strong and weak (wave) turbulence : HAT
- Artificial forcing, generally narrow-band in Fourier space : realistic only in strict HIT (e.g. good results from highest resolution DNS by Kaneda and coworkers)


Strain-dominated flow (hyperbolic)
Shear flow (linear)
Vorticity-dominated flow (elliptical)
2
1

Production ? vs. waves and neutral modes

- Homogeneous turbulence in the presence of a mean (base) flow with uniform mean gradients

$$U_i = A_{ij}(t)x_j + u_i, \qquad A_{ij} = S_{ij} + \epsilon_{ijn}W_j$$

  "RDT" (Batchelor, etc), Craya (1958), Rogallo (1981), come-back to linear stability, Craik, Bayly (1986), etc.

- "production" by $\mathbf{S}$: linear instability (eliptical and hyperbolical), possible *nonlinear* exponential growth (pure shear $S = W/2$). Homogeneity ??
- 'Pure' rotation $\mathbf{S} = 0 \longrightarrow$ projecting onto the rotating frame (Coriolis), inertial wave turbulence
- Similar cases without production : stable stratification with and without rotation, MHD with external magnetic field. Strong anisotropy (axisym) — with homogeneity — at all scales.

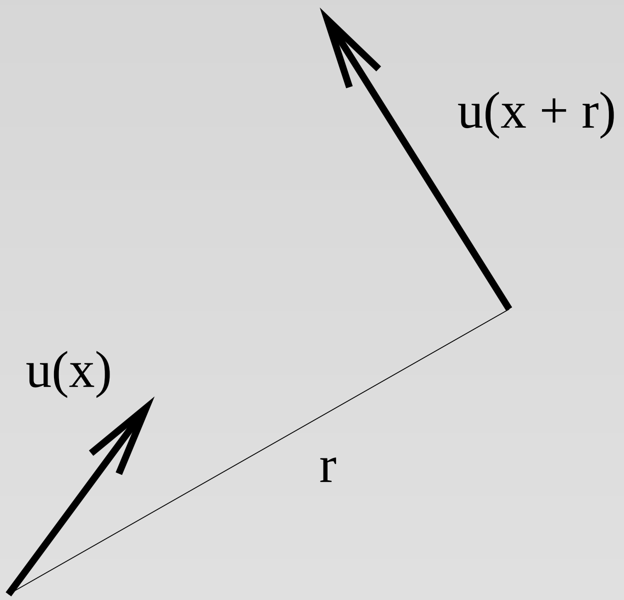
Applications

- Strong anisotropy *at all scales*, waves.
- Low Rossby (rapidly) rotating turbulence : inertial wave turbulence
- Stably stratified turbulence : toroidal cascade
- Others, MHD, weakly compressible HIT

Strong anisotropy

Anisotropic description

- ANISOTROPY/ inhomogeneity/ Intermitency
- structure functions or correlations, two-point : $R_{ij}(\boldsymbol{r}) = \langle u_i(\boldsymbol{x}) u_j(\boldsymbol{x}+\boldsymbol{r}) \rangle$



-) Single-point: componentality only

-) Two-point: directional anisotropy

- Low dimension parameterization, SO(3) symmetry group (Arad *et al.*,PRE,1999)

Anisotropic description. 3D Fourier space

- Anisotropic scalar (e. g. spherical harmonics) for both 'physical' and 'spectral'

$$\frac{1}{2}R_{ii}(\boldsymbol{r}) \quad \rightarrow \quad \frac{1}{2}\hat{R}_{ii}(\boldsymbol{k}) = e(\boldsymbol{k})$$

$$\sum r_n^m(r)Y_n^m(\theta_r,\phi_r) \quad \rightarrow \quad \sum \varphi_n^m(k)Y_n^m(\theta_k,\phi_k)$$

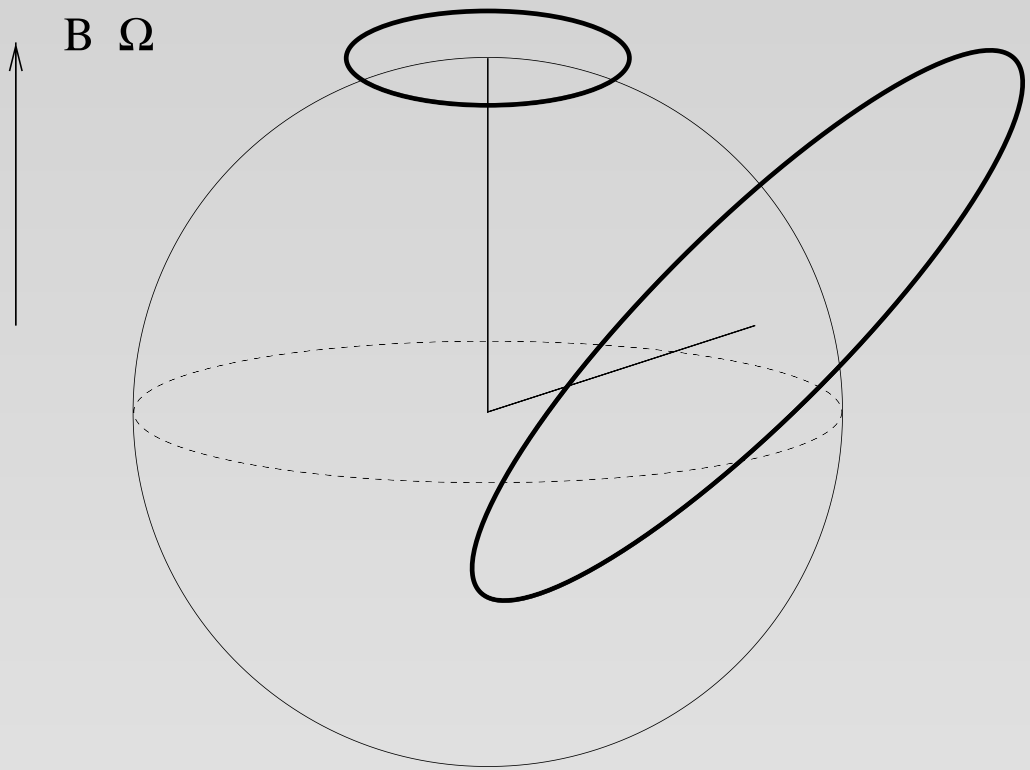
Avoiding a 'schizophrenic' viewpoint ! (Cambon & Teissèdre 1985, CRAS Paris)

- A trace-deviator decomposition restricted to solenoidal space

$$\hat{R}_{ij} = \underbrace{\underbrace{U(k)P_{ij}}_{\text{isotropic}} + \underbrace{\mathcal{E}(\boldsymbol{k})P_{ij}}_{\text{directional}}}_{eP} + \underbrace{\Re\left(Z(\boldsymbol{k})N_iN_j\right)}_{\text{polarization}}.$$

(Cambon & Jacquin, JFM, 1989), $P_{ij} = \delta_{ij} - \frac{k_ik_i}{k^2}$, $\boldsymbol{N}$ 'helical mode'. Helicity ?

Rotating turbulence, MHD simplified case, $\boldsymbol{k} \perp \hat{\boldsymbol{u}}$



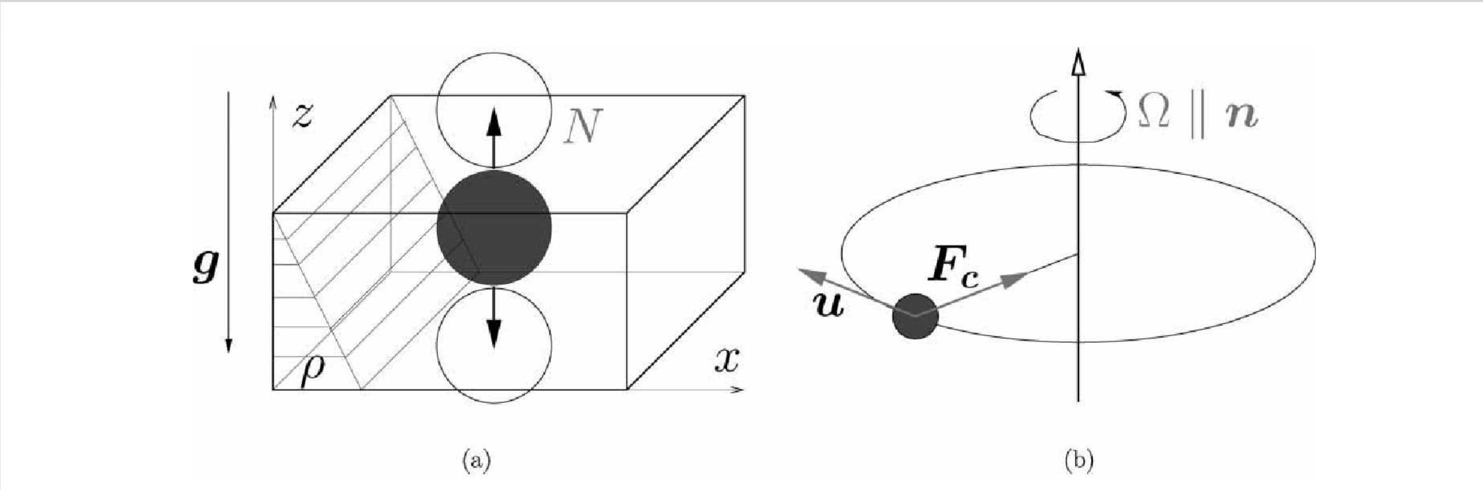
Stable stratification and rotation

Incompressible N-S with Coriolis + buoyancy $\boldsymbol{F} = 2\boldsymbol{\Omega} \times \boldsymbol{u} + b\boldsymbol{n}$

$\boldsymbol{n}$ local upstream vertical, Coriolis $f\boldsymbol{n} \times \boldsymbol{u}$

$\dot{b} + N^2 u_{||} - P_r \nu \nabla^2 b = 0$ (idem passive scalar with mean scalar gradient)

Cartoon of oscillations, towards waves



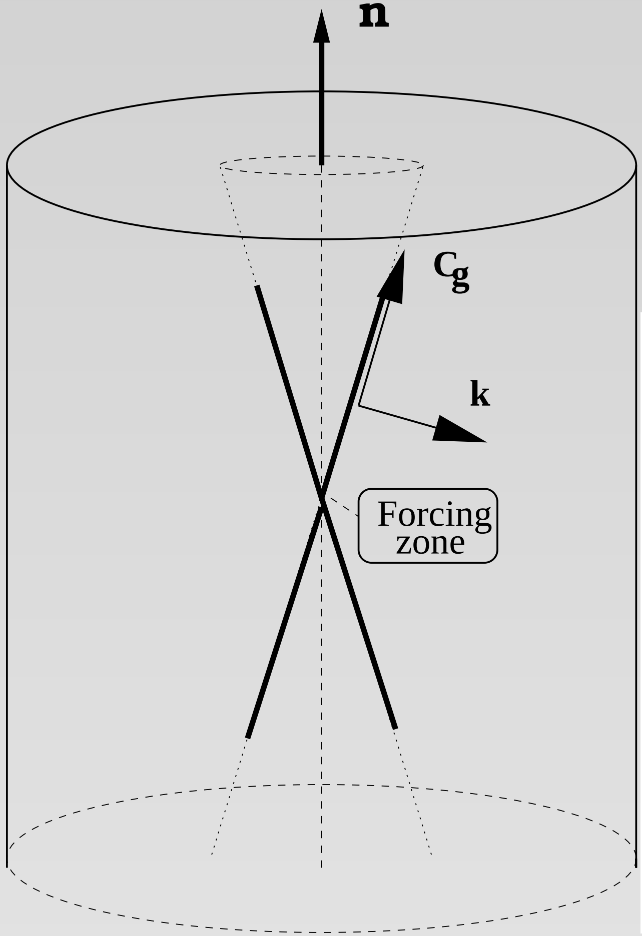
Two external parameters (frequencies) : N and f.

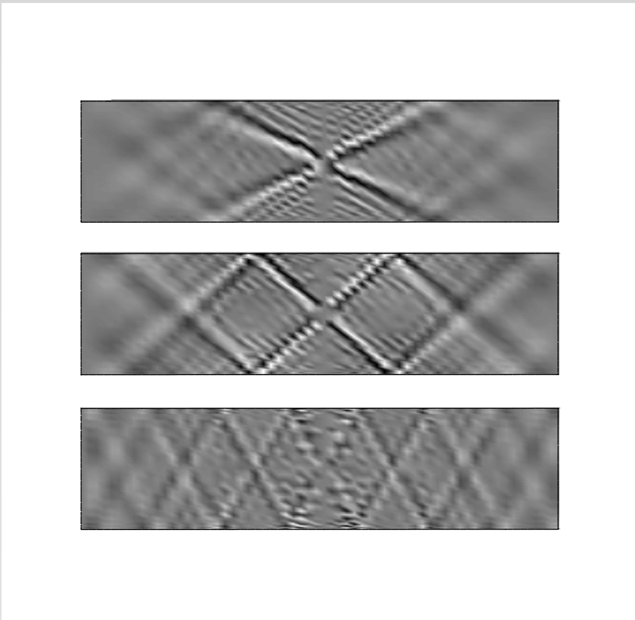
Eigenmodes decomposition

- incompressibility and pressure $\longrightarrow$ 3D Fourier space
  $(\boldsymbol{u}, b)(\boldsymbol{x}, t) =$
  $$\sum e^{\imath \boldsymbol{k}\cdot\boldsymbol{x}} \left(\underbrace{a_0 \boldsymbol{N}^{(0)}}_{\text{vortex (QG)}} + \underbrace{a_{+1}\boldsymbol{N}^{(1)} e^{\imath\sigma_k t} + a_{-1}\boldsymbol{N}^{(-1)} e^{-\imath\sigma_k t}}_{\text{wave (AG)}} \right)$$
- Dispersion law $\sigma_k = \sqrt{N^2 \sin^2\theta + f^2\cos^2\theta}$
- Linear dynamics: slow amplitudes $a_{0,\pm 1}$ are constant. Nonlinear case
- Advantages $\boldsymbol{k}\cdot\hat{\boldsymbol{u}} = 0$, five $(u_1, u_2, u_3, b, p) \longrightarrow$ three (a_0, a_{+1}, a_{-1}).

ref. Cambon et al., Bartello, Smith & Waleffe, Morinishi, Kaneda ...etc

Wave aspects





Mc Ewan (1967), Mowbray & Rarity (1967), Godeferd & Lollini, JFM (1999)

Rotation, phase mixing: Linear and nonlinear

- Relevance of linear solution depends on the *order and type* of statistical correlations

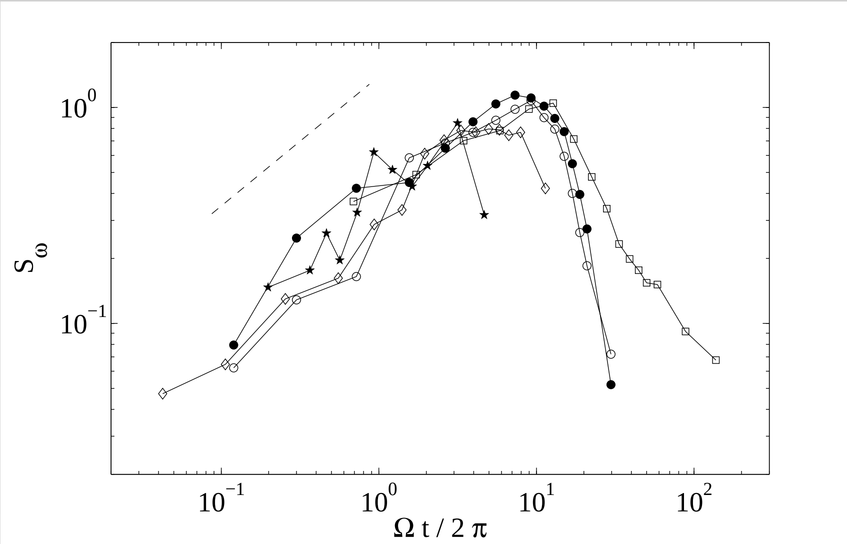
  -) doubles: 2 point 1 time: $e^{\imath\sigma_k t}$, $e^{-\imath\sigma_k t}$

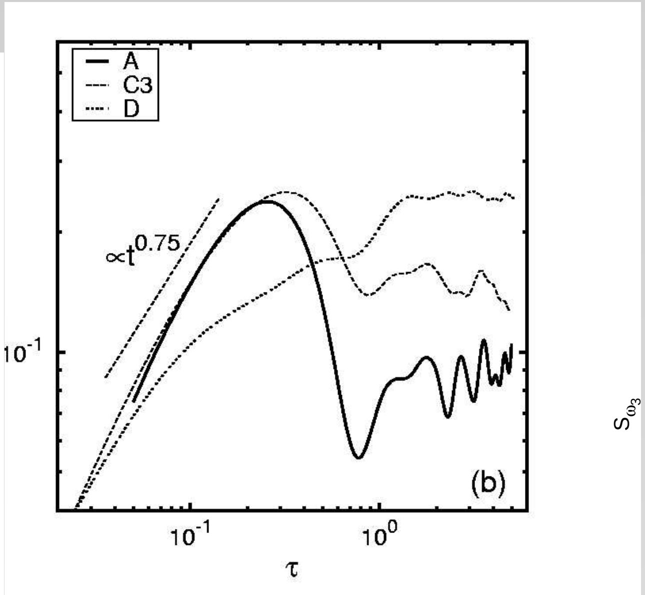
  -) doubles: 2 point 2 time: $e^{\imath\sigma_k (t \pm t')}$

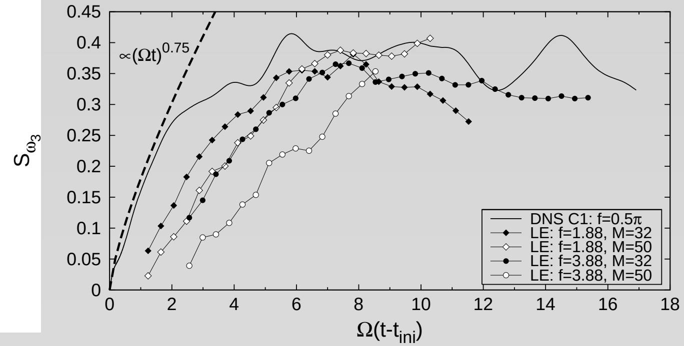
  -) triples: 3 point 1 time: $e^{\imath(\pm\sigma_k \pm\sigma_p \pm\sigma_q)t} \longrightarrow$ nonlinear $\cdots$

$$\langle \omega_3^3 \rangle = \sum \int \exp[\imath f t(\cos\theta_k + s' \cos\theta_p + s'' \cos\theta_q] S(\boldsymbol{k}, \boldsymbol{p}, \epsilon t) d^3\boldsymbol{p} d^3\boldsymbol{k}$$

($\cos\theta_k = k_3/k$) Need for initial triple correlations at THREE point. Many other correlations.







Cyclonic / anticyclonic asymmetry: (64^3 LES ?) Bartello et al. (1994), Morize et al. (2005), Gence & Frick (2001), Staplehurst et al. (2007), van Bokhoven et al. (2006). No need for centrifugal inst.

$$\frac{d}{dt}\langle\omega_3^3\rangle = 6\Omega\langle\frac{\partial u_3}{\partial x_3}\omega_3^2\rangle + \text{4th-order, viscous}$$

Exact spectral equations for nonlinear theory

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) e(k, \cos\theta, t) = T^{(e)}(k, \cos\theta, t)$$

Poincaré transformation $\hat{u}_i(\boldsymbol{k}, t) = \sum_{s=\pm 1} N_i(s\boldsymbol{k}) e^{\imath s f t \cos\theta} a_s(\boldsymbol{k}, \epsilon t)$,

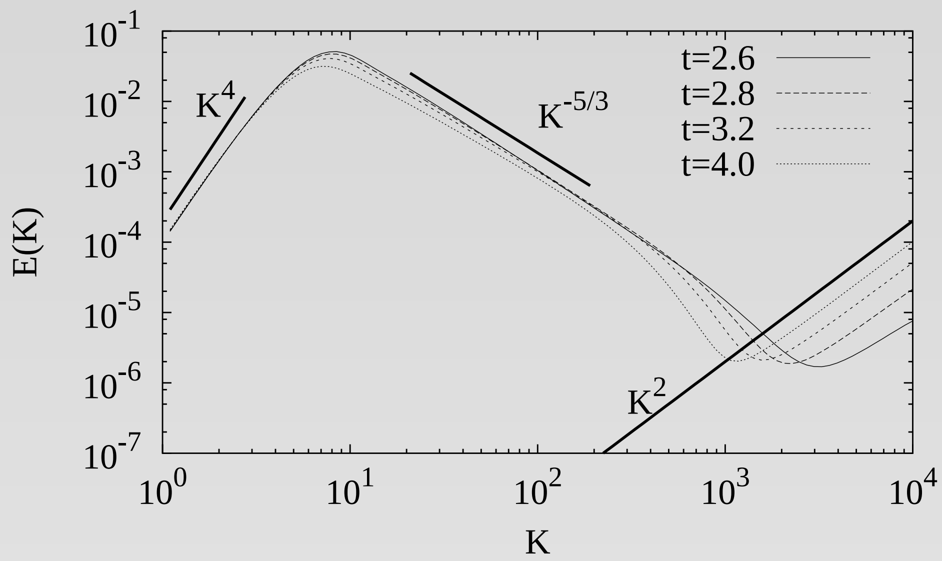
$$T^{(e)} = \int_{t_0}^{t} \sum_{s,s',s''=\pm 1} \int_{p+q=k} e^{\imath f (t-t')(s\cos\theta_k + s'\cos\theta_p + s''\cos\theta_q)}$$

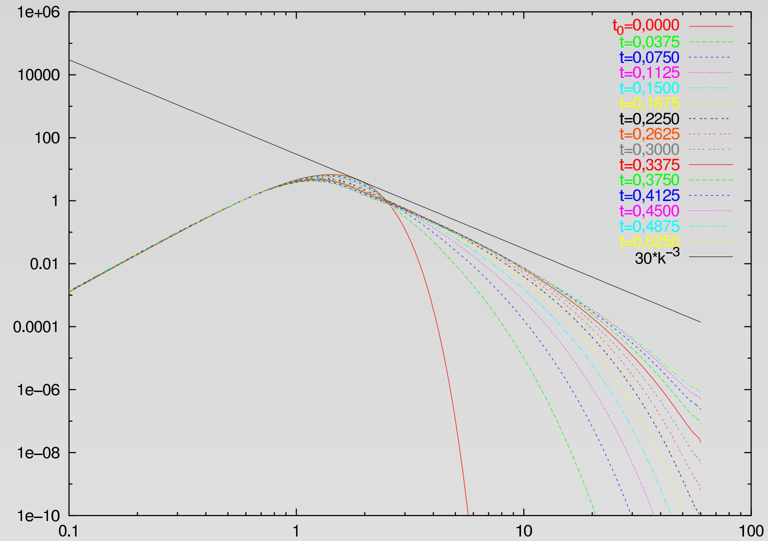
$$S_{ss's''}(\boldsymbol{k}, \boldsymbol{p}, t, \epsilon t') d^3\boldsymbol{p} dt',$$

with exact tradic conservation $\dot{a}_s(\boldsymbol{k}) = (s'p - s''q) G_{kpq} a^*_{s'}(\boldsymbol{p}) a^*_{s''}(\boldsymbol{q})$, possibly restricted to resonant triads.

Results. NONLINEAR statistical theory

- From classical EDQNM (isotropic, no rotation, Orszag 1970, Bos & Bertoglio, 2006) ...

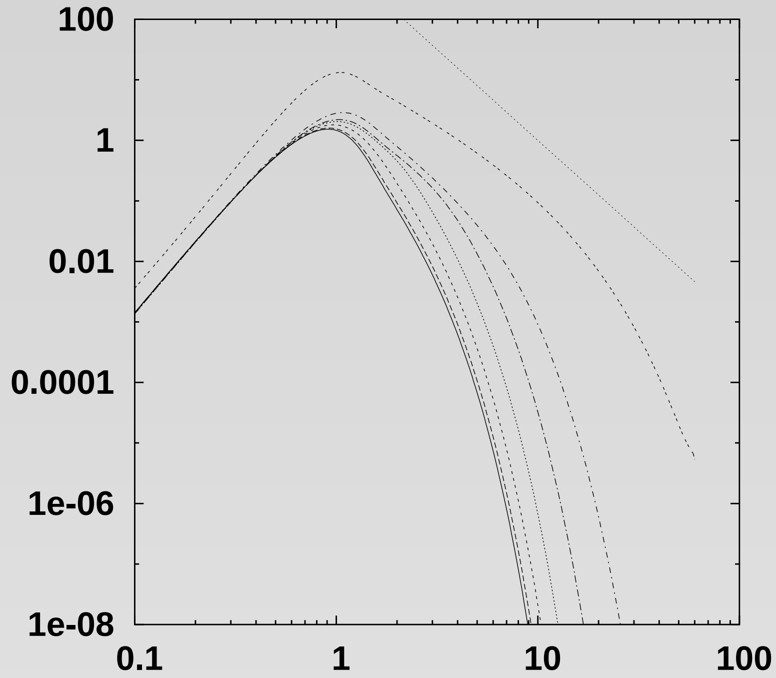




- ... to EDQNM3 $\longrightarrow$ (A) QNM energy equation (Bellet *et al.*, JFM, 2006)

Angle-dependent spectrum

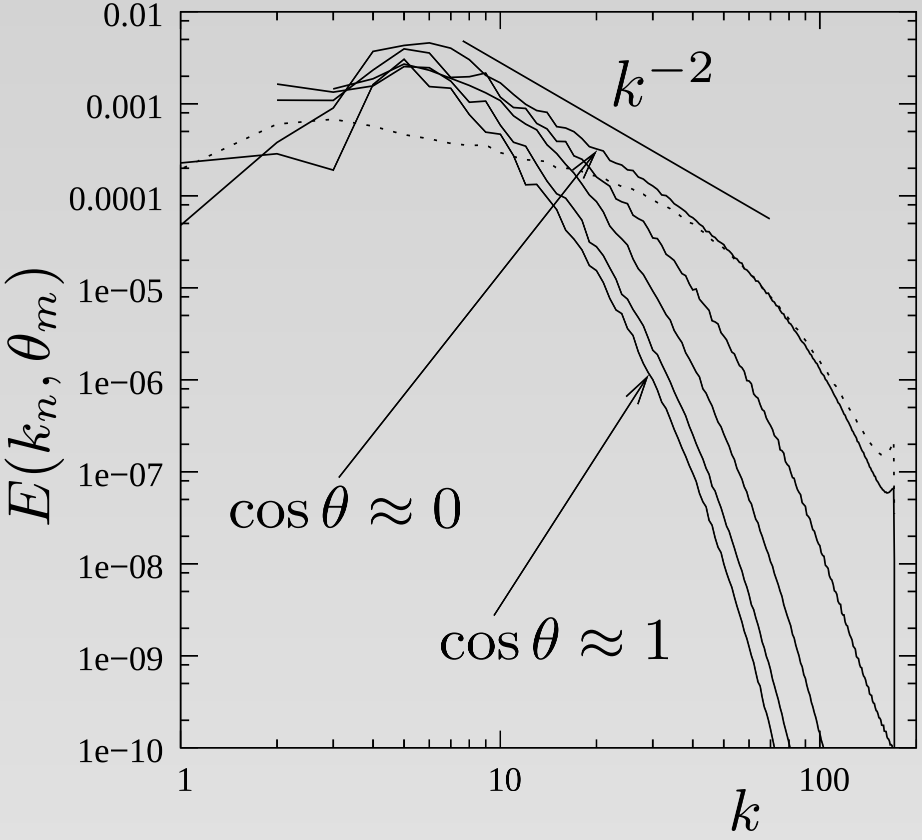
- Isotropy breaking by spectral transfer $T^{(e)}(\boldsymbol{k})$: directional anisotropy:

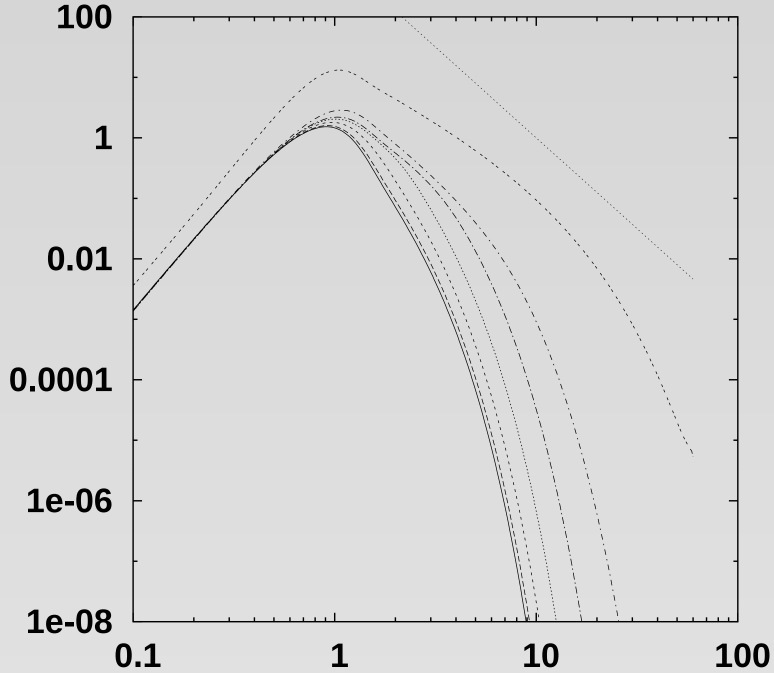


$$4\pi k^2 e(\boldsymbol{k}, t_f) = 4\pi k^2 e(k, \underbrace{\cos\theta}_{k_\parallel/k}, t_f)$$

- Spherical averaging $\longrightarrow E(k, t_f)$, prefactor $E \sim \frac{\Omega}{t} k^{-3}$, not 2D !

AQNM and DNS





512^3 DNS by Liechtenstein *et al.*, JOT, 2005

Inertial wave-turbulence, resonant interactions, 2D or not 2D

- *Low dimension* of active manifolds : overestimated in forced ? DNS/LES ?
  'TRUE' 2D embedded in 3D : a DIRAC singularity

$$E(k) \sim f^2 k^{-3}, \qquad e(k_\perp, k_\parallel) = \underbrace{\frac{E(k_\perp)}{2\pi k_\perp}}_{\sim f^2 k_\perp^{-4}} \delta(k_\parallel)$$

- integral singularity from theoretical wave-turbulence

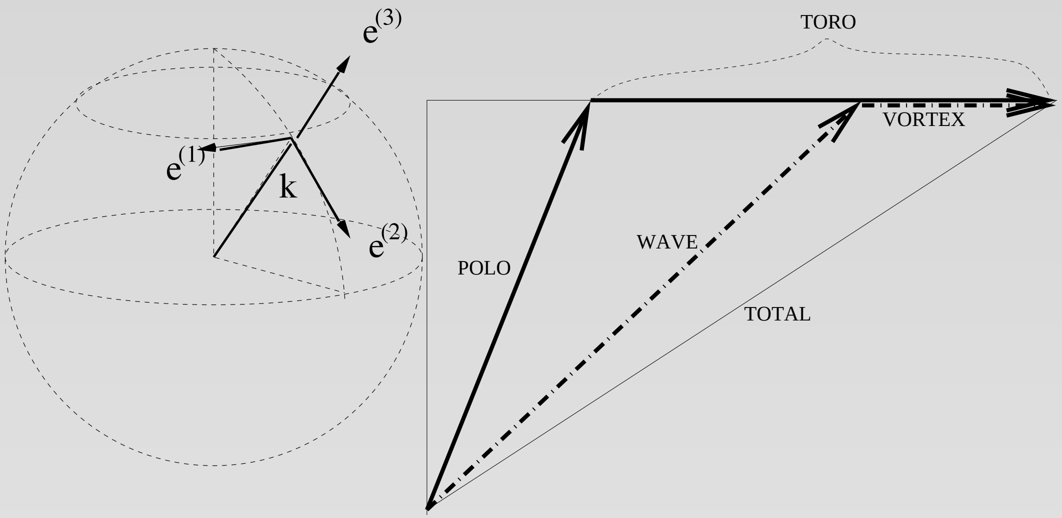
$$e(k_\perp k_\parallel) \sim k_\parallel^{-1/2} k_\perp^{-7/2} = k^{-4} x^{-1/2} \quad \text{Galtier 2002}$$

$$e(k_\perp k_\parallel) \sim k_0^{-1/2} k_\parallel^{-1/2} k_\perp^{-3} = k_0^{-1/2} k^{-7/2} x^{-1/2} \quad \text{CRG 2004}$$

$$E(k) \sim \frac{f}{t} k^{-3}, \quad e(k, x \sim 0) \sim k^{-4} \quad \text{BGSC-2006}$$

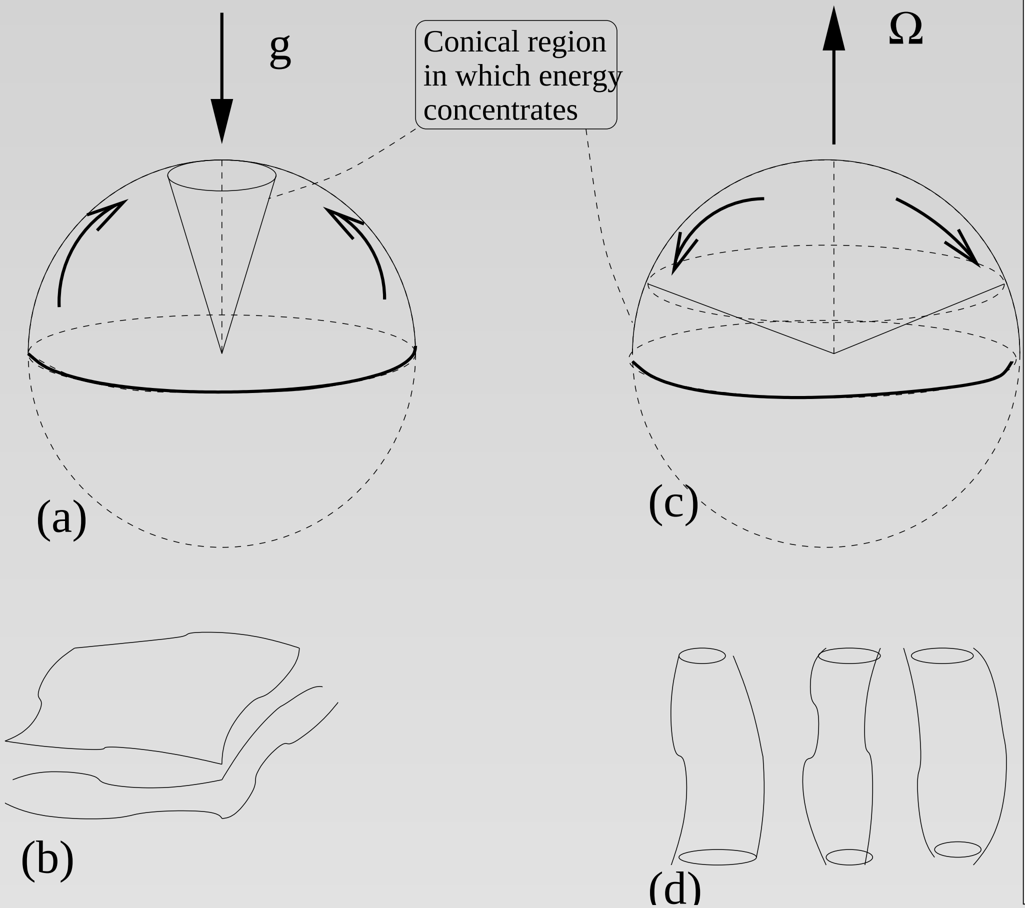
Reintroducing stratification : ANTI 2D and toroidal (strong) cascade

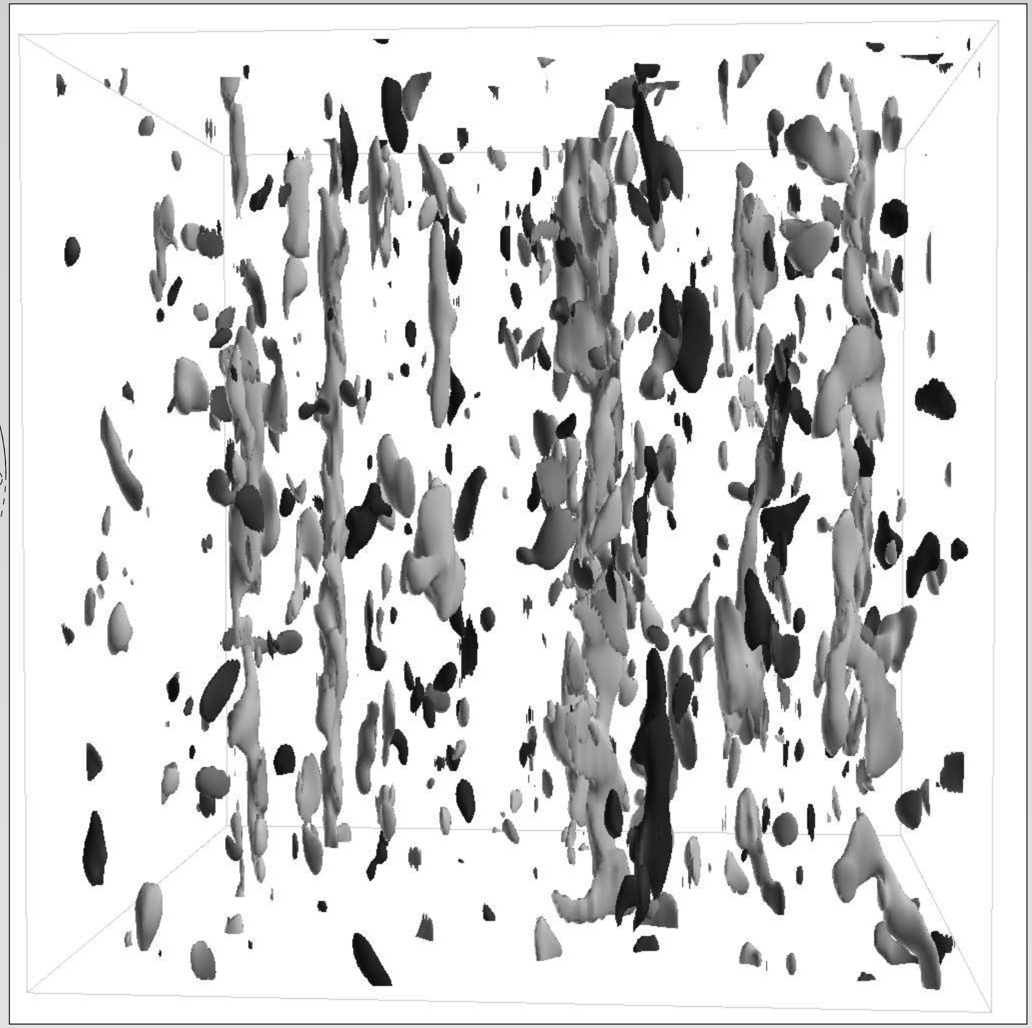
Geometric aspects of eigenmodes

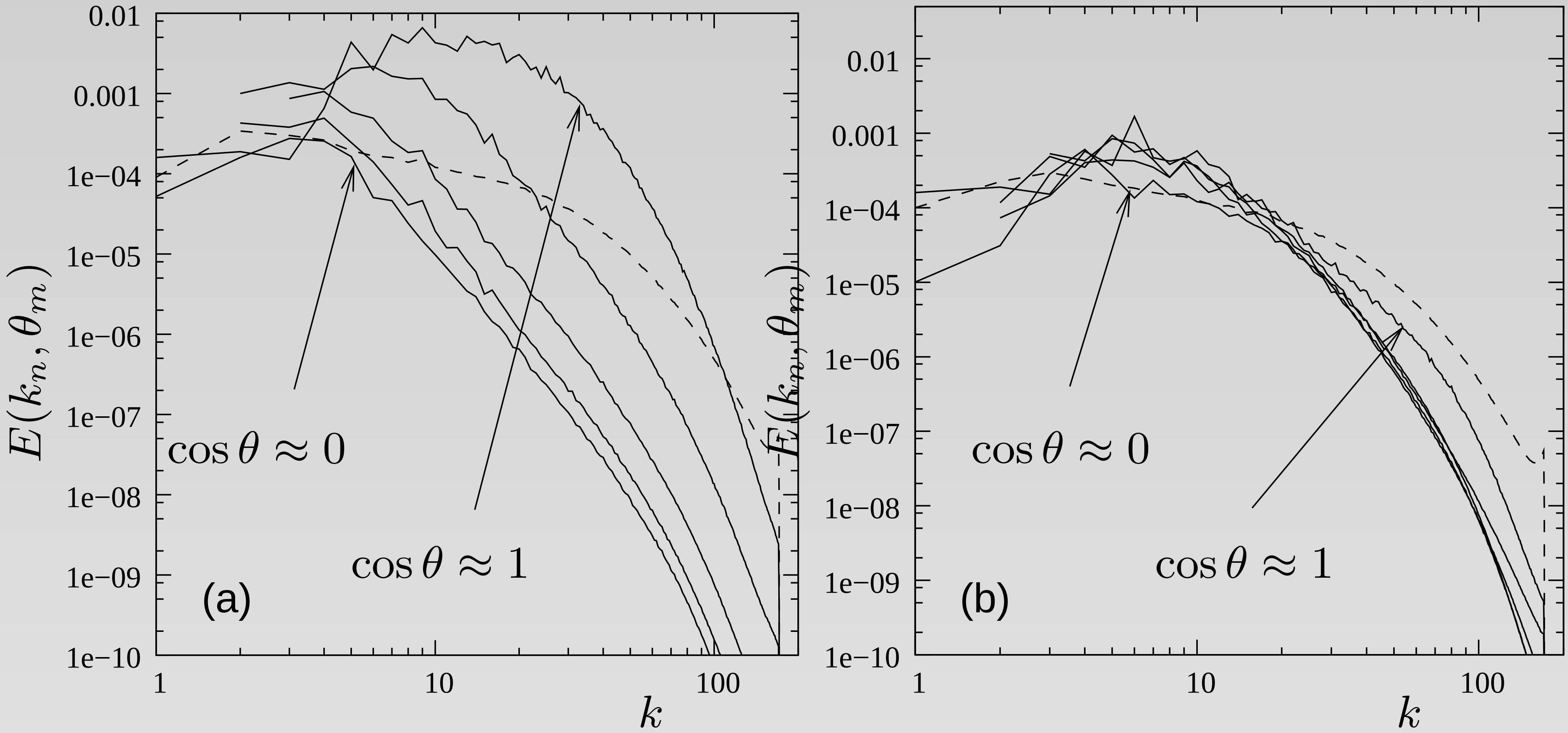


Toroidal/poloidal

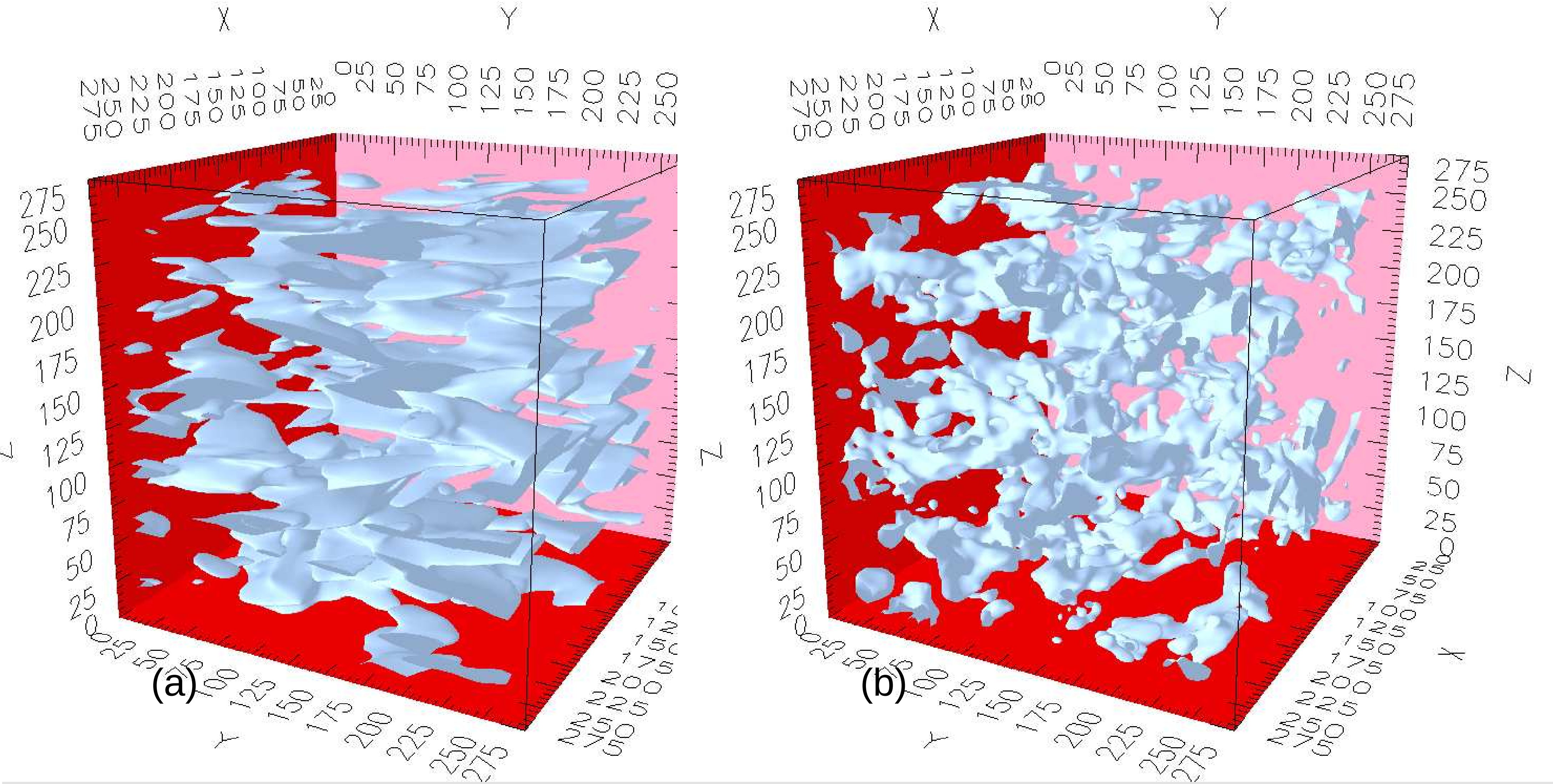
(Craya-Herring, standard) and “Vortex/wave” ($f/N-$ depending)


g
Conical region
in which energy
concentrates
Ω
(a)
(c)
(b)
(d)





Angle-dependent toroidal and poloidal modes (Liechtenstein, 2006)


(a)
(b)

Pure stratification: scaling arguments

- 2D or not 2D ? Charney (1971), Lilly (1983), Lindborg (1999) using third order structure functions from observations.
- Froude numbers, horizontal and vertical, $Fr_h = U/(NL_h)$, $Fr_v = U/(NL_v)$, $L_h \gg L_v$.
- $L_v \sim U/N$ ("zig-zag" instability ? Billant & Chomaz 1999, 2002) $\longrightarrow Fr_v \sim 1$ (to contrast with Riley et al. 1981 ?)
- Proposed scaling (Linborg 2006) $E_{hh}(k_h) = C_1 \varepsilon_k^{2/3} k_h^{-5/3}$, $E_p(k_h) \sim C_2 \epsilon_p \epsilon_K^{-1/3} k_h^{-5/3}$, $E_{vv} \sim N^2 k_v^{-3}$.
- $Ri \sim 1/4$ threshold vs. $Fr^2 Re$ scaling (Riley) ?, rediscovery of polo-toro decomposition (Brethouwer, Linborg, Billant, Chomaz.)

Pure stratification: Generalized Lin equations

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) e^{(tor)} = T^{(tor)} \tag{1}$$

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) e^{(w)} = T^{(w)} \tag{2}$$

$$\left(\frac{\partial}{\partial t} + 2\nu k^2 + 2\imath N\frac{k_\perp}{k}\right) Z' = T^{(z')} \tag{3}$$

Energy spectra $e^{(tor),(pol),(pot)}(k_\perp, k_{||})$, imbalance deviator Z',
$\Re Z' = e^{(pol)} - e^{(pot)}$, $e^{(w)} = e^{(pol)} + e^{(pot)}$.
A lot of information can be generated, vs. second and third-order structure functions.

The toroidal cascade. Why not 2D ?

- Why the toroidal component only ? $\boldsymbol{e}^{(1)} \cdot \ldots \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{u} + \boldsymbol{\nabla}\left(p + \frac{u^2}{2}\right) - b\boldsymbol{n}$,
  $\dot{u}^{(1)} + \boldsymbol{e}^{(1)} \cdot \sum_{p+q=k} (\hat{\boldsymbol{\omega}}(\boldsymbol{p}) \times \hat{\boldsymbol{u}}(\boldsymbol{q})) = 0$,
  $\hat{\boldsymbol{u}} = u^{(1)}\boldsymbol{e}^{(1)} + u^{(2)}\boldsymbol{e}^{(2)}$, $\hat{\boldsymbol{\omega}} = \imath k \left(u^{(1)}\boldsymbol{e}^{(2)} - u^{(2)}\boldsymbol{e}^{(1)}\right)$
- Following Kraichnan and Waleffe (1992, 1993): stability of a single triad

$$\dot{u}_k^{(1)} = (p_\perp^2 - q_\perp^2) G u_p^{(1)*} u_q^{(1)*}, \tag{4}$$

$$\dot{u}_p^{(1)} = (q_\perp^2 - k_\perp^2) G u_q^{(1)*} u_k^{(1)*}, \tag{5}$$

$$\dot{u}_q^{(1)} = (k_\perp^2 - p_\perp^2) G u_k^{(1)*} u_p^{(1)*}, \tag{6}$$

- quasi 2D or not, reverse or direct cascade ? cylinder to cylinder, shell to shell, angle to angle : very rich and various morphology ...

Analogy with an 'Euler problem'

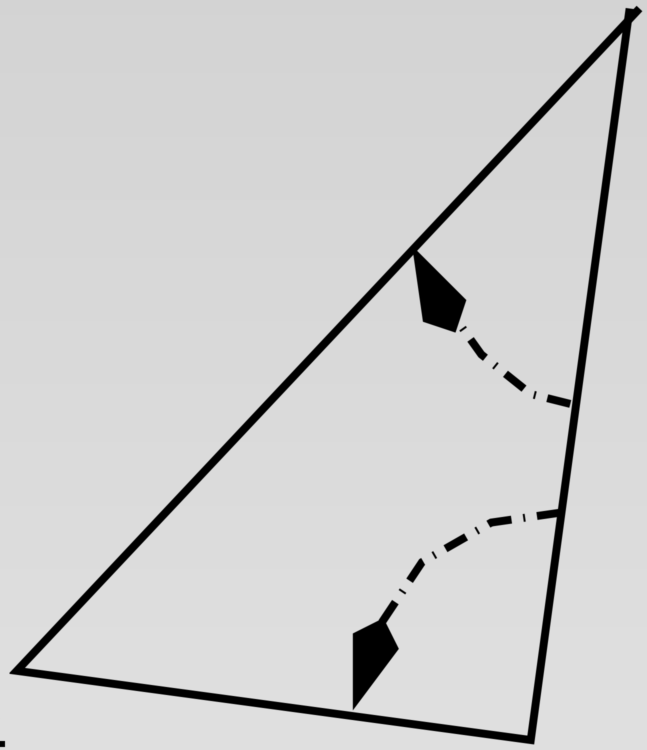
- The solid in its principle axes of inertia

$$I_1\dot{\Omega}_1 = (I_2 - I_3)\Omega_2\Omega_3, \tag{7}$$

$$I_2\dot{\Omega}_2 = (I_3 - I_1)\Omega_3\Omega_1 \tag{8}$$

$$I_3\dot{\Omega}_3 = (I_1 - I_2)\Omega_1\Omega_2, \tag{9}$$

- Conservations laws, rot. kin. energy $(I_1\Omega_1^2 + \ldots) \longrightarrow$ kin. energy (triad), norm of the angular momentum $(I_1\Omega_1)^2 + \ldots \longrightarrow$ vertical enstrophy (triad)



- Instabilities, reverse interactions only.

Revisiting the QG cascade

- Detailed conservation of QG energy and potential enstrophy (linear ?)

$$\frac{k^2\sigma_k^2}{N^2}\xi^{(0)}\xi^{(0)*} = k_\perp^2 u^{(1)}u^{(1)*} + \left(\frac{f}{N}k_\parallel\right)^2 u^{(3)}u^{(3)*}$$

- Reworking on N^{000} (Bartello 1995)

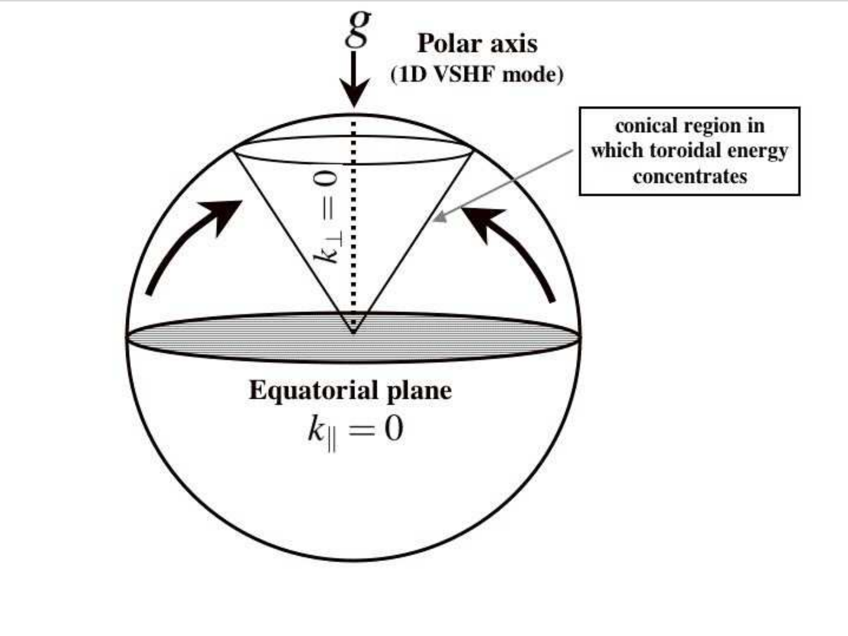
$$\dot{\xi}_k^{(0)} = (p^2\sigma_p^2 - q^2\sigma_q^2)G'\xi_p^{(0)*}\xi_q^{(0)*}, \quad (10)$$

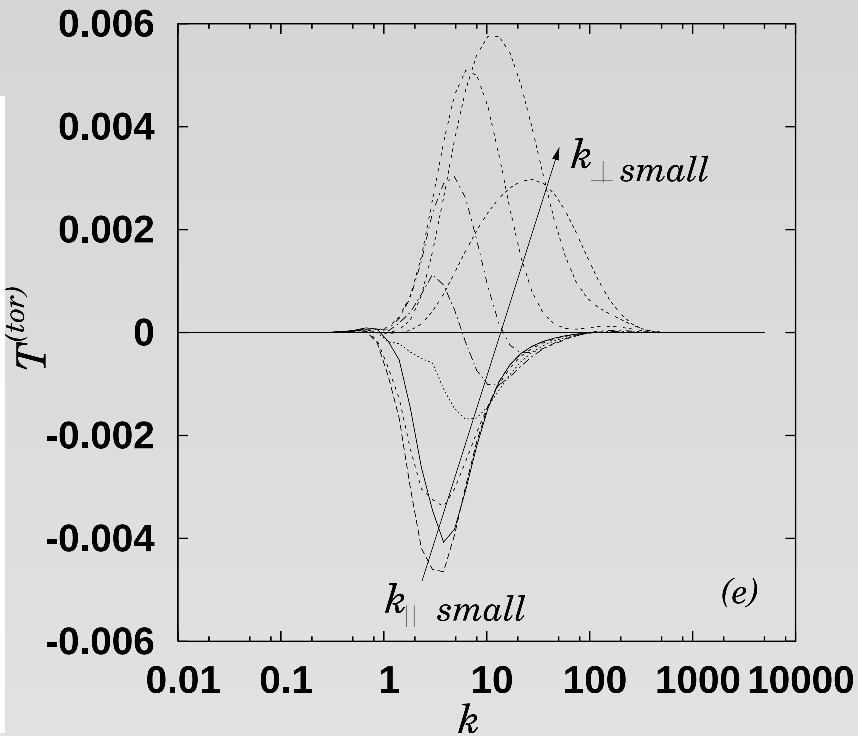
$$\dot{\xi}_p^{(0)} = (q^2\sigma_q^2 - k^2\sigma_k^2)G'\xi_q^{(0)*}\xi_k^{(0)*}, \quad (11)$$

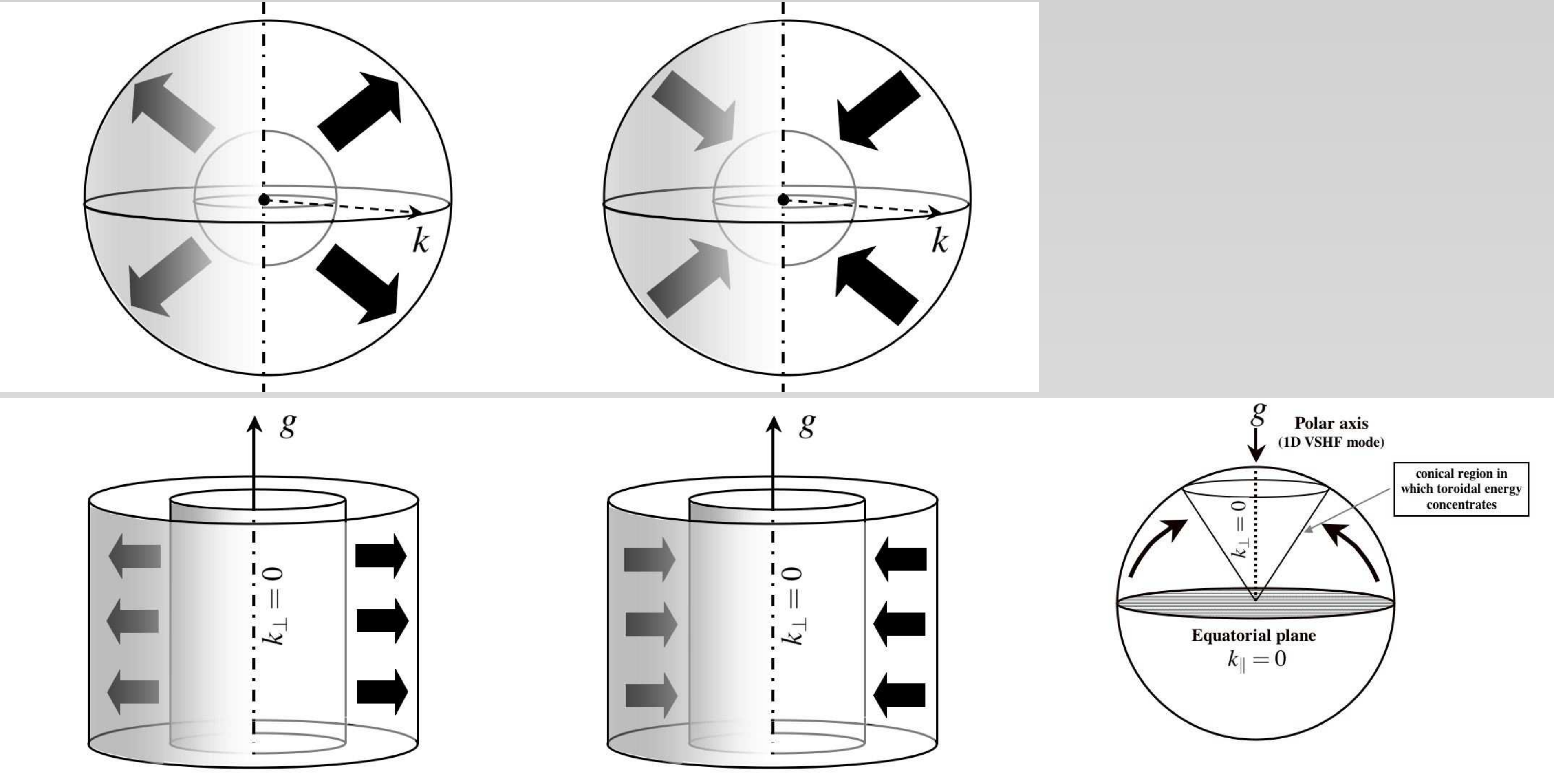
$$\dot{\xi}_q^{(0)} = (k^2\sigma_k^2 - p^2\sigma_p^2)G'\xi_k^{(0)*}\xi_p^{(0)*}, \quad (12)$$

- ‘linear’ (quadratic) limit ? Ertel theorem, flat isopycnes. Dual cascade or not, why ?

Toroidal main energy drain, ‘wave-released’ EDQNM2

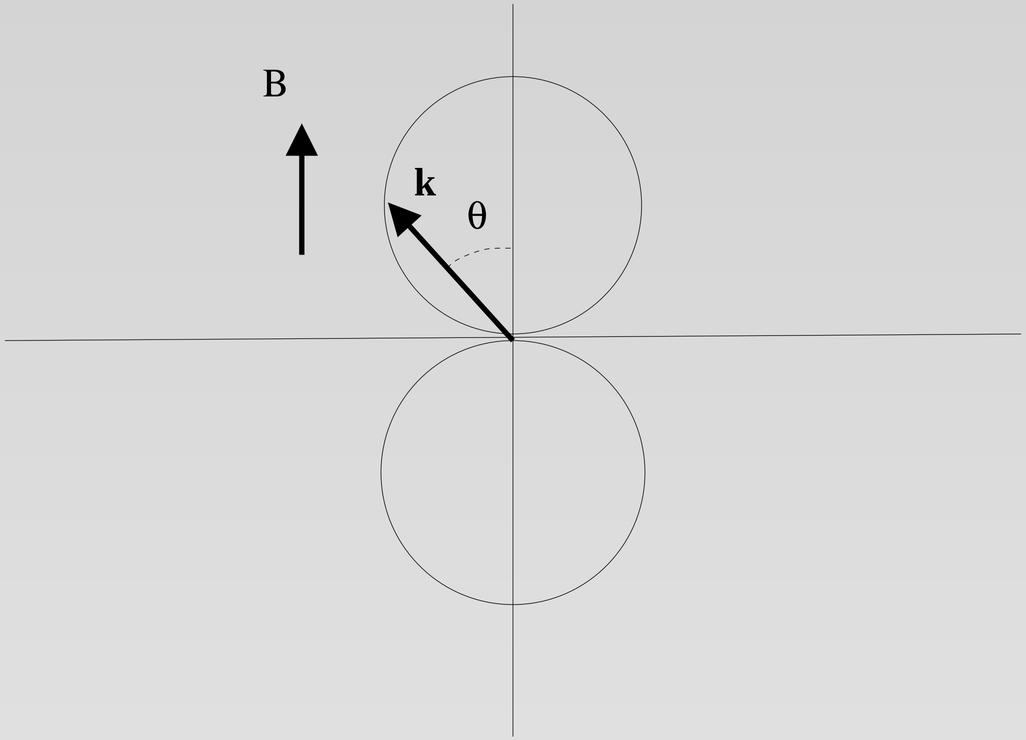






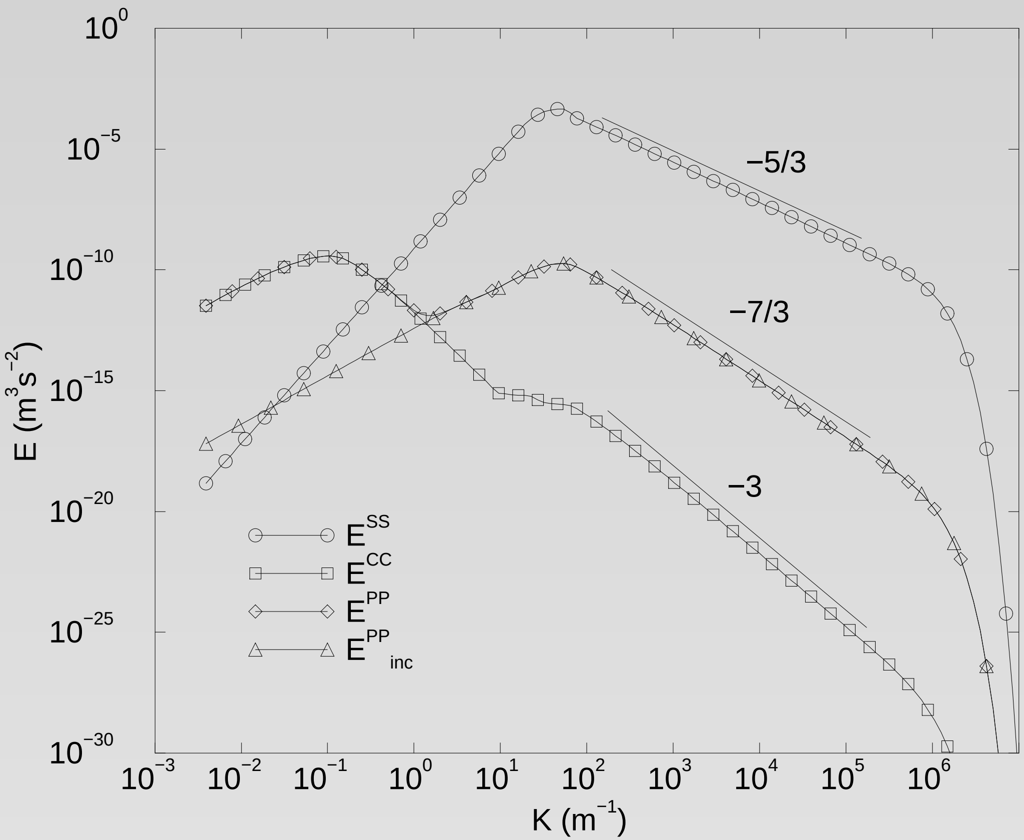
k
k
g
$k_\perp = 0$
g
$k_\perp = 0$
g
Polar axis
(1D VSHF mode)
conical region in which toroidal energy concentrates
$k_\perp = 0$
Equatorial plane
$k_\parallel = 0$

Coexistence of weak and strong turbulence: MHD



Alfven-wave turbulence competing with strong turbulence with additional Joule dissipation effect (Moffat 1967, Alboussière, etc)

Coexistence of weak and strong turbulence: Aeroacoustics

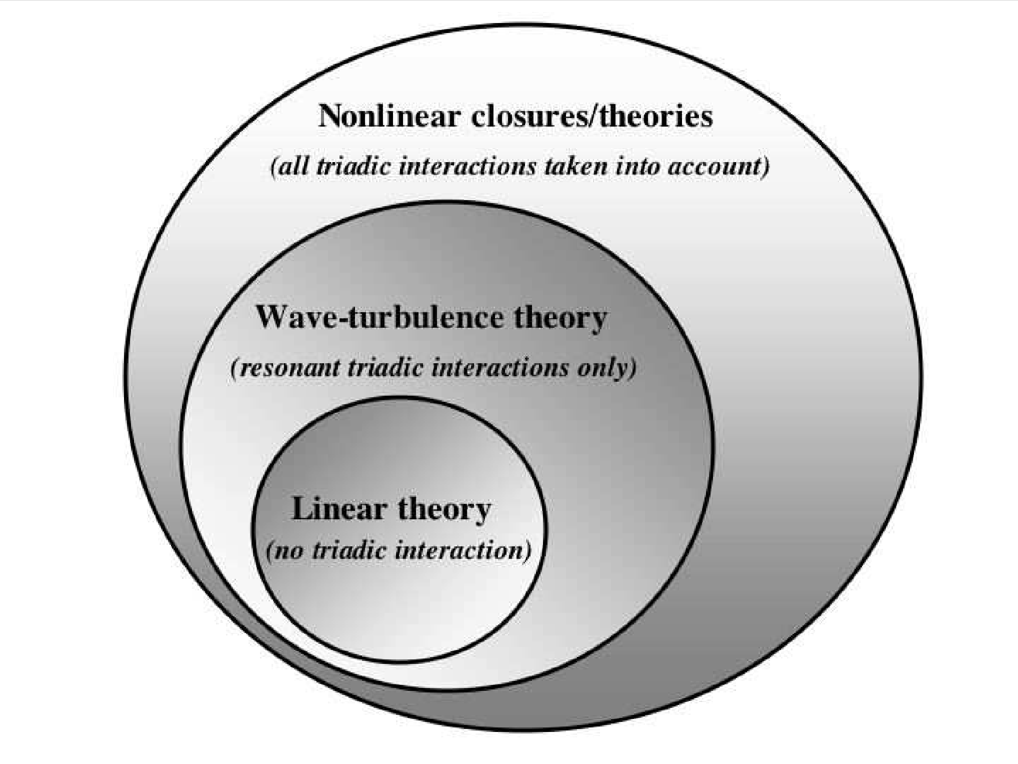


Quasi-isentropic THI at low Mach number, Fauchet et al. (1997).

Some concluding comments, open issues

1. What about structure functions and $4/5$ equation for strong anisotropy at all scale ? Interest for vorticity, potential vorticity too
2. Anisotropic cascade : $\langle (\delta u_{\|})^3 \rangle$ versus $T^i(k_{\|}, k_{\perp})$?
   -) Exact Lin-type equations
   -) A systematic way to construct triadic correlations, consistently with detailed conservation laws
   -) Need for improved ED for strong turbulence (Wouter-EDQNM, TFM, LRA ?)
3. Competition between waves and vortices to organise Lagrangian diffusion (also plasmas ?)
4. A better link between physical and spectral space ? (ANISO)

Towards a theory of axisymmetric weak+strong turbulence



- Work remain to be done for toroidal, QG, + main waves interactions (catalytic ?)
- Toro-polo with Elsasser variables, MHD, plasmas
- Not to forget DNS/LES with dedicated post-processing (Marenostrum project, etc)