Active vs Passive Scalar Turbulence

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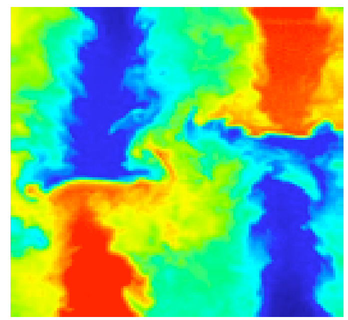

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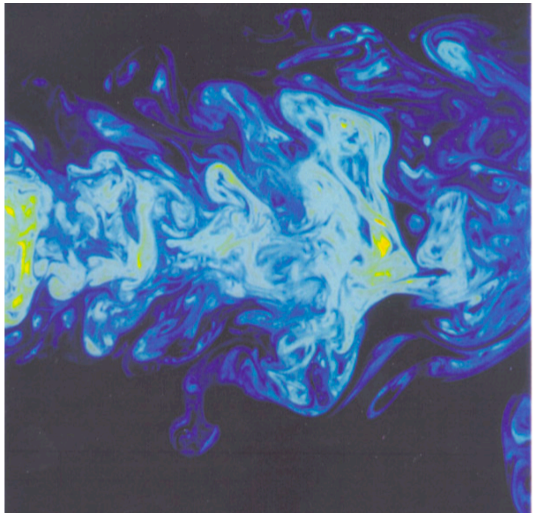
Cargese Aug 2007

Outline

- Transport of Fields by turbulence: examples
- Eulerian & Lagrangian view of transport
- Passive scalars: a paradigm for universality and intermittency
  - Energy cascade & dissipation
  - Intermittency & universality
- Active vs Passive scalars: 4 examples at comparison
  - Turbulent convection (bsq)
  - Magnetohydrodynamics (mhd)
  - Eckman turbulence (eck)
  - Surface Quasi-Geostrophic turbulence (sqg)
- Conclusions

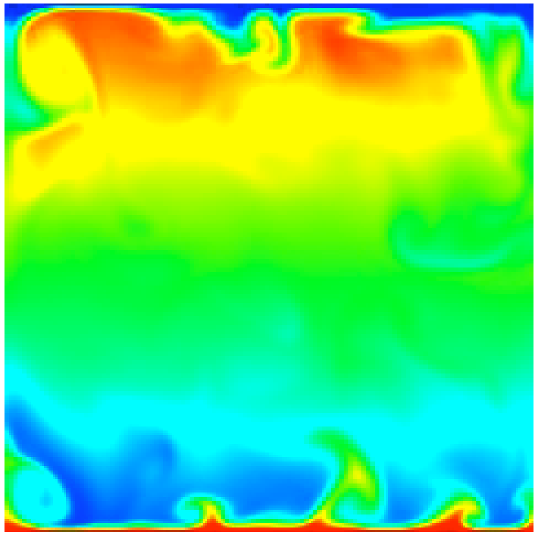
Fields in turbulent flows



Colorant dye

Passive Fields

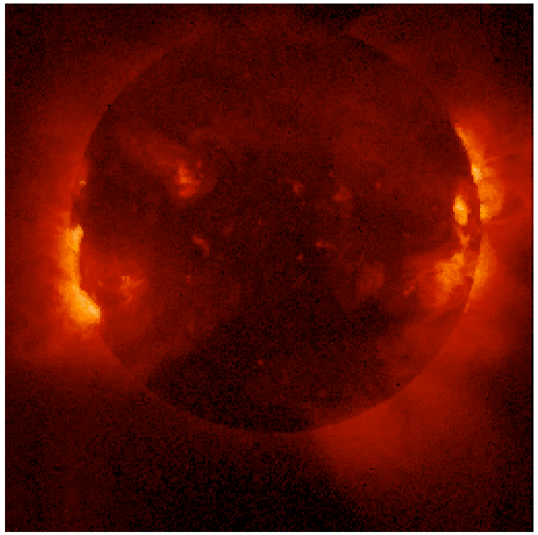
Smoke



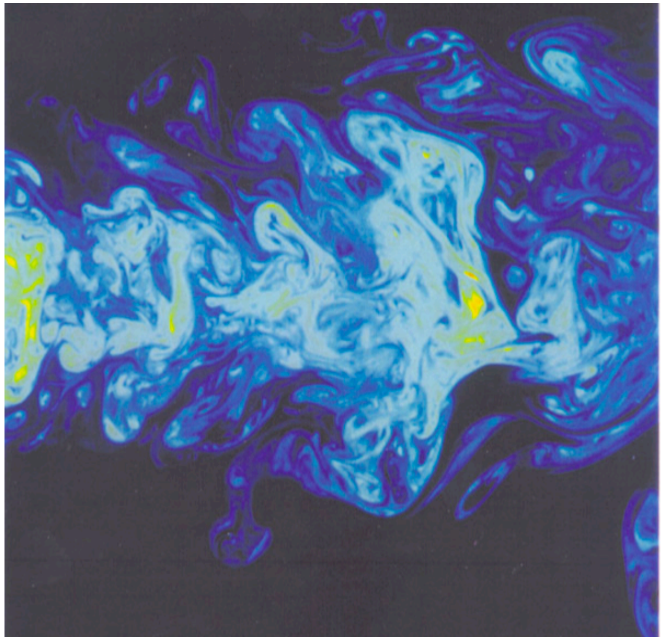
Temperature

Active Fields

Magnetic field



Passive Scalar Fields



The velocity field is given and not modified by the transported field

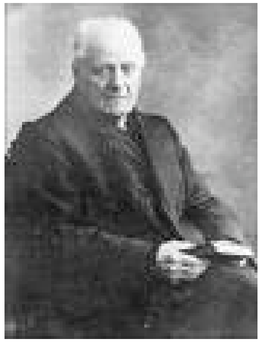
$$\boxed{\partial_t c + \boldsymbol{v} \cdot \nabla c = \kappa \Delta c + F_c}$$

Phenomenology similar to NS turbulence

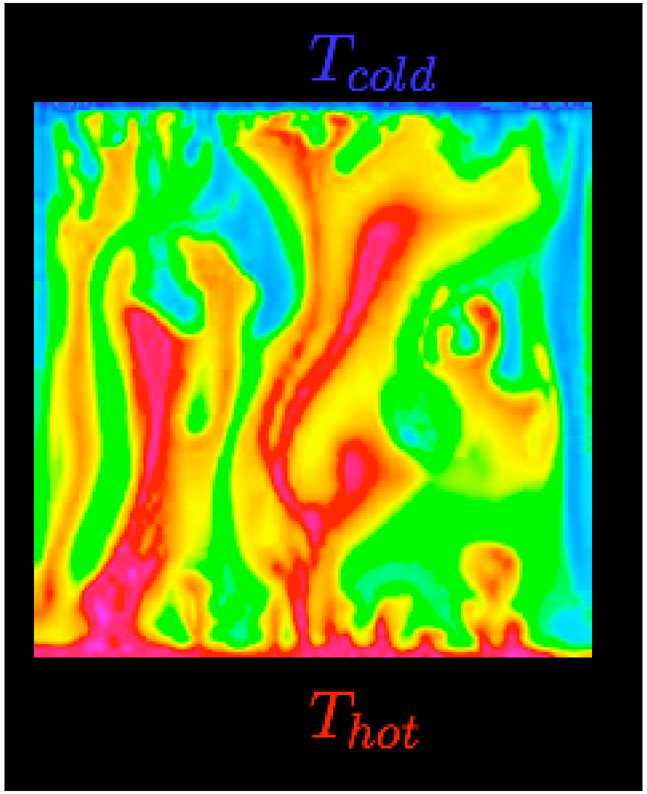
- Cascade towards the small scales
- Finite energy dissipation *(dissipative anomaly)*
- Intermittency of the small scales

Goal: understanding dynamics & statistics as a function of v:
Is there universality with respect to the forcing F_c?

Active Scalar Fields



J. Boussinesq



Hele-Show flow in 2d

Rayleigh-Benard Convection

(Bizon et al. *Chaos* **7**, 1 (1997))

E.g. Temperature is transported by

$$\partial_t a + \boldsymbol{v} \cdot \boldsymbol{\nabla} a = \kappa \Delta a + F_a$$

And modifies the velocity through the Boussinesq (Buoyancy) term

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = \nu \Delta \boldsymbol{v} - \boldsymbol{\nabla} P - \beta g a$$

Active Scalar Fields

In general: we consider active scalar fields acting on the velocity field through local forces

$$\partial_t a + \boldsymbol{v} \cdot \boldsymbol{\nabla} a = \kappa \Delta a + F_a \qquad \textit{active field}$$

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = -\boldsymbol{\nabla} p + \nu \Delta \boldsymbol{v} + \boxed{\mathcal{F}(a, \boldsymbol{\nabla} a, \dots)}$$

$$\partial_t c + \boldsymbol{v} \cdot \boldsymbol{\nabla} c = \kappa \Delta c + F_c \qquad \textit{passive field}$$

The rank of the problem passes from the linear world of passive fields to the nonlinear one, much more similar to the problem of NS-turbulence itself

Active Scalar Fields

Another class of active scalar fields

$$\partial_t a + \boldsymbol{v} \cdot \boldsymbol{\nabla} a = \kappa \Delta a + F_a \qquad \textit{active field}$$
$$v_i(\boldsymbol{x}, t) = \int d\boldsymbol{y}\, \Gamma_i[\boldsymbol{x}, \boldsymbol{y}]\, a(\boldsymbol{y}, t)$$

$$\partial_t c + \boldsymbol{v} \cdot \boldsymbol{\nabla} c = \kappa \Delta c + F_c \qquad \textit{passive field}$$

Scalar and velocity fields are functionally related

The specific form of the kernel Γ depends on the case under considerations.
Equations of this kind frequently occur in geophysical fluid mechanics

Active Scalar Fields

Example: vorticity in NS 2d

$$\boldsymbol{v} = (v_1, v_2) \qquad a = \boldsymbol{\nabla} \times \boldsymbol{v} = \partial_1 v_2 - \partial_2 v_1$$

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = -\boldsymbol{\nabla} p + \nu \Delta \boldsymbol{v} \qquad \partial_t a + \boldsymbol{v} \cdot \boldsymbol{\nabla} a = \nu \Delta a$$

Soap film turbulence
Y. Amarouchene and H. Kellay
Phys. Rev. Lett. **93**, 214504

Given the vorticity, the velocity is

obtained inverting the *curl*

$$v_i(\boldsymbol{x}, t) = \int d\boldsymbol{y}\, \Gamma_i[\boldsymbol{x}, \boldsymbol{y}]\, a(\boldsymbol{y}, t)$$

with

$$\Gamma_i[\boldsymbol{x}, \boldsymbol{y}] = -(2\pi)^{-1}\, \epsilon_{ij} \partial_j \log|\boldsymbol{x} - \boldsymbol{y}|$$

More on 2d turbulence: lecture by A. Lanotte

Two words about the velocity

We are interested in the transport of fields and particles in velocity fields v(x,t) which are typically characterized by scaling properties

$$\boxed{\delta_r v \approx r^h}$$

h=1 smooth (differentiable) flows

h<1 rough (Holder continuous) flows

Note: In rough flows strange things happen Lagrangian paths are not unique

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{v}(\boldsymbol{X},t) + \text{"}noise\text{"}$$

$$\begin{aligned} \boldsymbol{R} &= \boldsymbol{X}_1 - \boldsymbol{X}_2 \\ \frac{dR}{dt} &= v(\boldsymbol{X}_2 + R, t) - v(X_2, t) = \delta_r v \sim R^h \end{aligned}$$

$$\frac{dR}{dt} = R^h$$

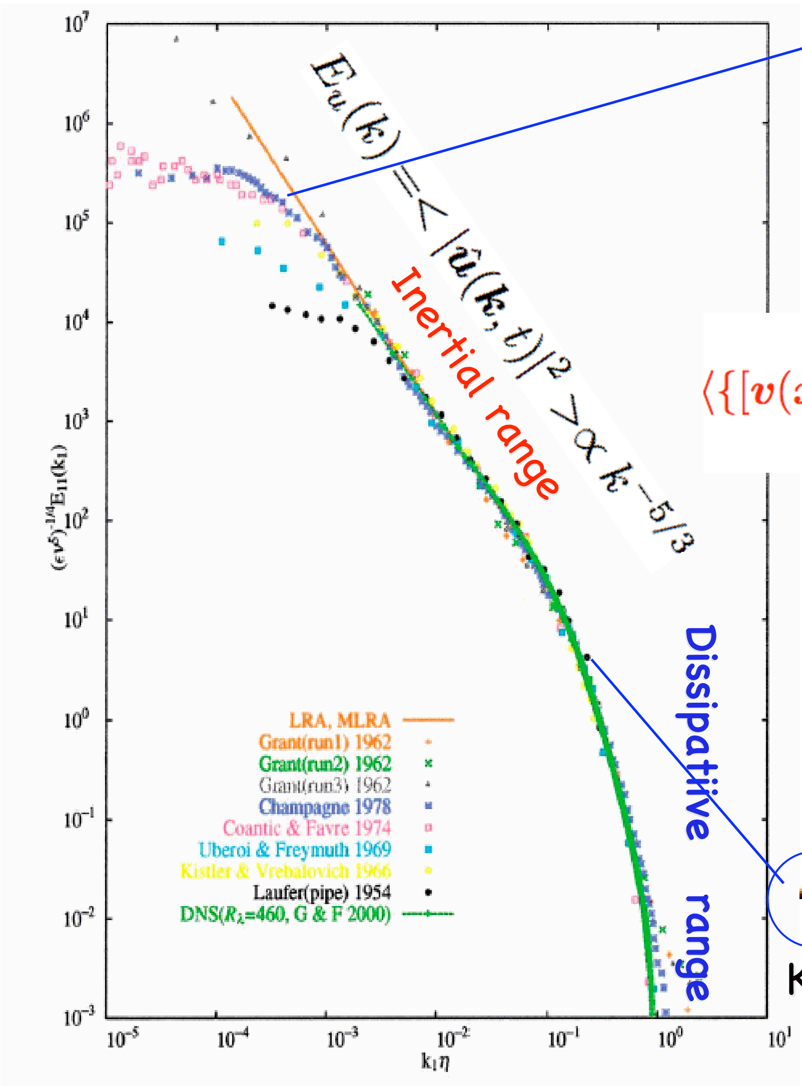
$$R(t) = [R^{1-h}(0) + t]^{\frac{1}{1-h}} \underbrace{\longrightarrow}_{t\to\infty} t$$

If h<1 dependence on the initial conditions is quikly wiped out

Rough flows are not unusual!!

3d Turbulence

Energy spectrum



L

integral scale

η

Kolmogorov scale

A.N.K. Kolmogorov

$$\langle\{[\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{v}(\boldsymbol{x},t)]\cdot\hat{\boldsymbol{r}}\}^2\rangle \sim \begin{cases} r^2 & r<\eta \\ r^{2/3} & \eta<r<L \\ const & r>L \end{cases}$$

smooth

rough

"4/5" law

$$\langle(\delta_r \boldsymbol{v}\cdot\hat{\boldsymbol{r}})^3\rangle = -\frac{4}{5}(\epsilon_v r)$$

$$\boxed{\text{K41 } \langle(\delta_r \boldsymbol{v}\cdot\hat{\boldsymbol{r}})^p\rangle \propto (\epsilon_v r)^{p/3}}$$

In reality more complex: intermittency etc..

Eulerian & Lagrangian description

$$\theta = c, a$$

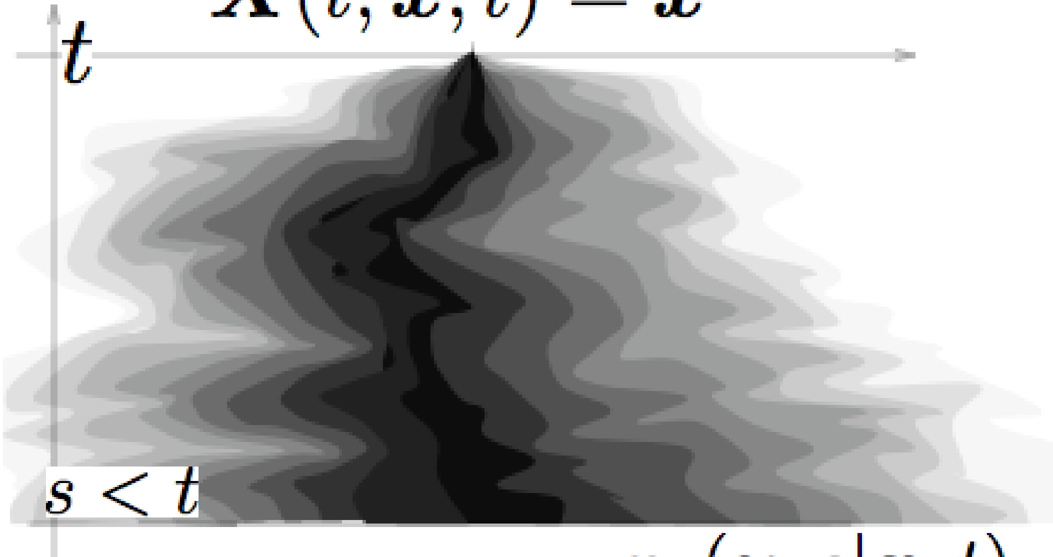
Eulerian (Fokker-Planck eq.)

$$\partial_t \theta + \boldsymbol{v} \cdot \boldsymbol{\nabla}\theta = \kappa \nabla^2 \theta + F_\theta$$

Lagrangian (Langevin eq.)

$$\mathrm{d}\boldsymbol{X}(s) = \boldsymbol{v}(\boldsymbol{X}(s), s)\mathrm{d}s + \sqrt{2\kappa}\mathrm{d}\boldsymbol{\beta}(s)$$

$$\boldsymbol{X}(t;\boldsymbol{x},t) = \boldsymbol{x}$$



Backward in time propagator $p_{\boldsymbol{v}}(\boldsymbol{y}, s|\boldsymbol{x}, t)$

$$\dot{\vartheta}(s) = F_\theta(\boldsymbol{X}(s;\boldsymbol{x},t), s)$$

$$\theta(\boldsymbol{x},t) = \langle \vartheta(t) \rangle_\beta = \left\langle \int_{-\infty}^{t} ds F_\theta(\boldsymbol{X}(s;\boldsymbol{x},t), s) \right\rangle_\beta = \int_{-\infty}^{t} ds \int d\boldsymbol{y}\, p_{\boldsymbol{v}}(\boldsymbol{y}, s|\boldsymbol{x}, t)\, F_\theta(\boldsymbol{y}, s)$$

X(s;x,t)- fluid trajectory ending in **x** at time t $\quad$ F_θ- scalar injection source

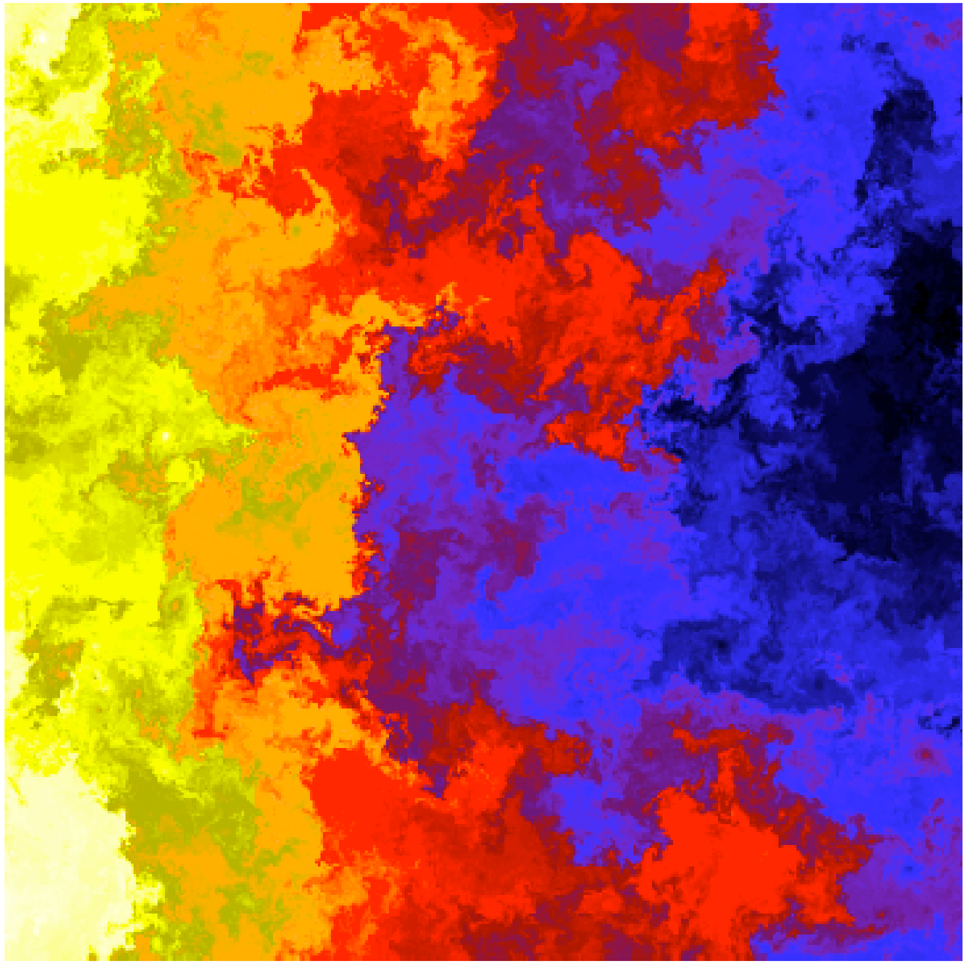
See U. Frisch, A. Mazzino, A. Noullez, and M. Vergassola, Phys. Fluids **11**, 2178 (1999)

Passive scalars

Review of basic phenomenology & results

Eulerian & Lagrangian views on

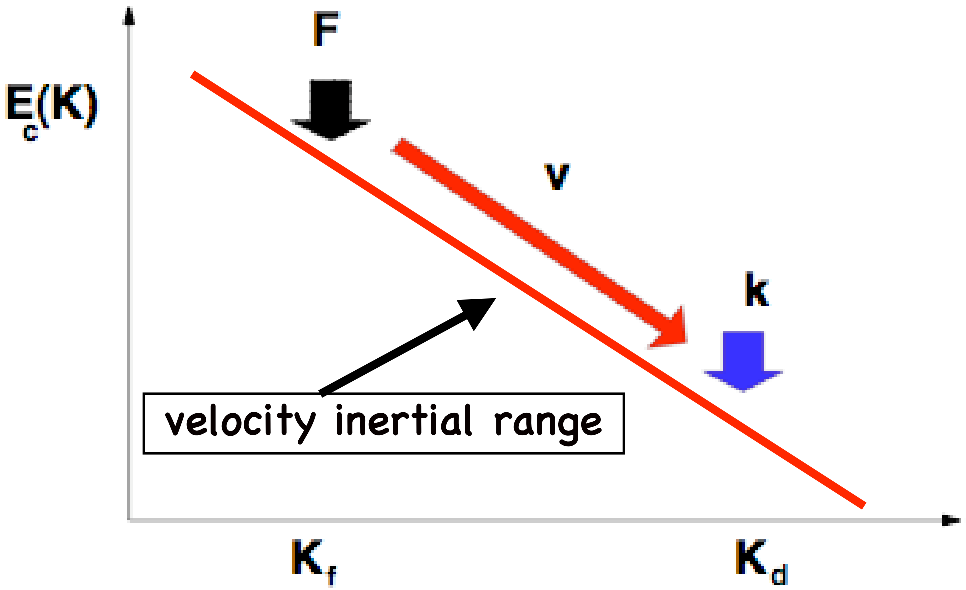
- Energy cascade
- Finite energy dissipation
- Intermittency
- universality



Celani & Vergassola, Phys. Rev. Lett. **86**, 424 (2001)

Cascade & Dissipation: Eulerian

$$\partial_t c + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{c} = \kappa \Delta c + F_c$$



$$\boxed{\frac{1}{2}\frac{d}{dt}\int d\boldsymbol{x}\, c^2(\boldsymbol{x},t) = < cF_c > - \kappa < |\nabla c|^2 > = F_0 - \epsilon_c \approx 0}$$

v distributes scalar energy among the modes with a constant flux F_0

Dissipation due to diffusion at small scales at a rate $\varepsilon_c \approx F_0$ also for $\kappa \to 0$ (dissipative anomaly)

Thanks to $\varepsilon_c \approx F_0$ a statistically steady state is reached

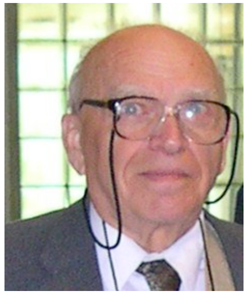
Cascade & Dissipation: Eulerian

Understanding scalar statistics: increments $\delta_r c = c(r,t) - c(0,t)$ & spectrum $E_c(k)$

Yaglom Relation (exact!) ==> $\langle \delta_r v (\delta_r c)^2 \rangle = -\frac{4}{3}\epsilon_c r$

Assuming K41 ==> $\delta_r v \sim (\epsilon_v r)^{1/3}$

Dimensional arguments ==> $\delta_r c \sim \epsilon_c^{1/2} \epsilon_v^{-1/6} r^{1/3}$



A. Yaglom

Oboukhov-Corrsin Spectrum and 2[nd] order structure function

A.M. Oboukhov

$$E_c(k) \propto k^{-5/3} \Longleftrightarrow C_2(r) = \langle c(\boldsymbol{r},t)c(\boldsymbol{0},t)\rangle \approx A - Br^{2/3}$$

$$S_2(r) = \langle (c(\boldsymbol{r},t) - c(\boldsymbol{0},t))^2 \rangle = 2(C_2(r) - C_2(0)) \propto r^{2/3}$$

S. Corrsin

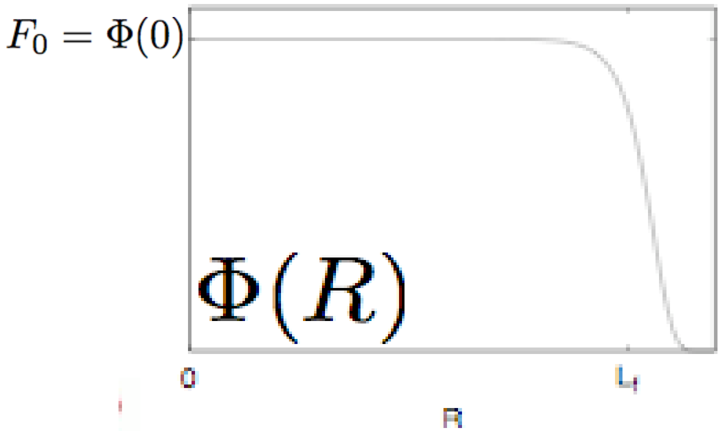
homogeneity, isotropy etc... are assumed

Cascade & Dissipation: Lagrangian

2nd order sf & spectrum from a Lagrangian viewpoint

$$c(\boldsymbol{x},t) = \int_{-\infty}^{t} ds\, d\boldsymbol{y}\, p_v(\boldsymbol{y},s|\boldsymbol{x},t)\, F_c(\boldsymbol{y},s)$$

$$C_2(r) = \langle c(\boldsymbol{r},t)c(0,t)\rangle_{F,v} = \iint_{-\infty}^{t} ds_1 ds_2 \iint d\boldsymbol{y}_1 d\boldsymbol{y}_2 \boxed{\langle p_v(\boldsymbol{y}_1,s_1|\boldsymbol{r},t)p_v(\boldsymbol{y}_2,s_2|\boldsymbol{0},t)\rangle_v} \boxed{\langle F_c(\boldsymbol{y}_1,s_1)F_c(\boldsymbol{y}_2,s_2)\rangle_F}$$



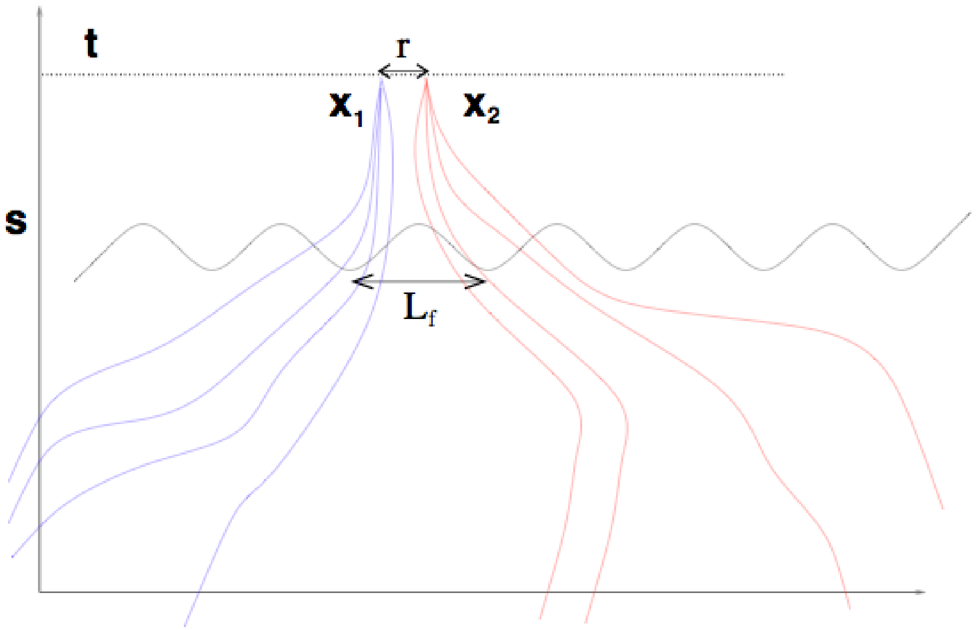
$$\langle F_c(\boldsymbol{y}_1,s_1)F_c(\boldsymbol{y}_2,s_2)\rangle_F = \delta(s_1-s_2)\Phi(R)$$

$$P_2(\boldsymbol{R},s|\boldsymbol{r},t) = \langle p_v(\boldsymbol{y}_1,s|\boldsymbol{r},t)p_v(\boldsymbol{y}_2,s|\boldsymbol{0},t)\rangle_v$$

$$C_2(r) = \int_{-\infty}^{t} ds \int d\boldsymbol{R}\, P_2(\boldsymbol{R},s|\boldsymbol{r},t)\, \Phi(R)$$

See U. Frisch, A. Mazzino, A. Noullez, and M. Vergassola, Phys. Fluids **11**, 2178 (1999)

Cascade & Dissipation: Lagrangian



Richardson diffusion

$$\frac{dR}{dt} = \delta_R v \approx R^{1/3}$$

$$R(t) \approx (R^{2/3}(0) + t)^{3/2} \to t^{3/2}$$

$$\Downarrow$$

$$\mathcal{T}(0, L_f) - \mathcal{T}(r, L_f) \sim r^{2/3}$$

$$S_2(r) = 2(C_2(0) - C_2(r))$$

$$\approx 2\Phi(0) \underbrace{\boxed{(\mathcal{T}(0, L_f) - \mathcal{T}(r, L_f))}}_{\mathcal{T}(0,r)} \propto r^{2/3}$$

$$C_2(r) = \int_{-\infty}^{t} ds \int d\boldsymbol{R}\, P_2(\boldsymbol{R}, s|\boldsymbol{r}, t)\, \Phi(R) =$$

$$\Phi(0) \int_{-\mathcal{T}(r,L_f)}^{t} ds \int d\boldsymbol{R}\, P_2(\boldsymbol{R}, s|\boldsymbol{r}, t) \approx \Phi(0) \boxed{\mathcal{T}(r, L_f)}$$

Time for the pair to (backward in time) separate from r to L_f

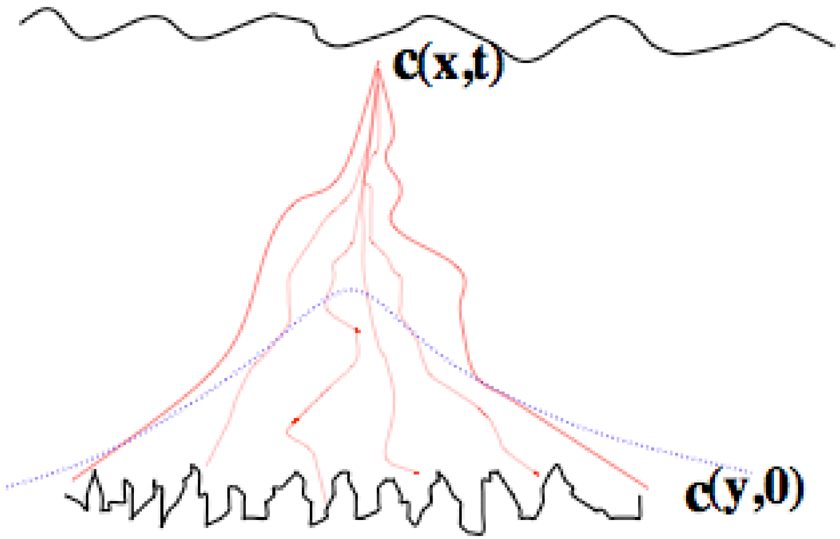
Cascade & Dissipation: Lagrangian

Observation: velocity roughness => non-uniqness of trajectories

$$R(t) \approx (R^{2/3}(0) + t)^{3/2} \to t^{3/2}$$

non-uniquness of trajectories ==> finite dissipation of energy also for $\kappa \to 0$

(lagrangian origin of the dissipative anomaly)



in the absence of forcing the initial condition is smoothed as time goes on due to the presence of many paths

$$\frac{d}{dt} c^2(t) = -\epsilon_c$$

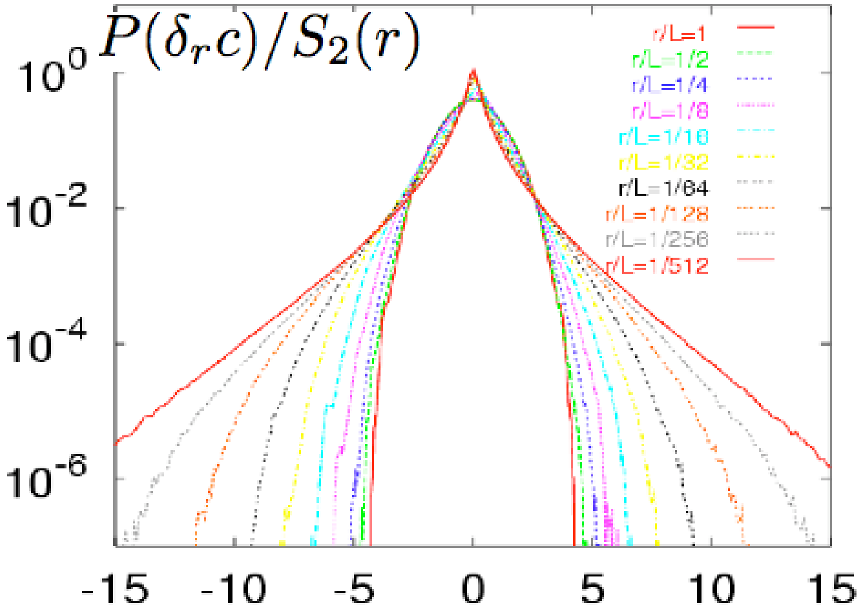
$$\kappa \to 0 \qquad \epsilon_c \neq 0$$

In smooth flows $\delta_r v \approx r$ paths are unique and $\kappa > 0$ is needed to dissipate

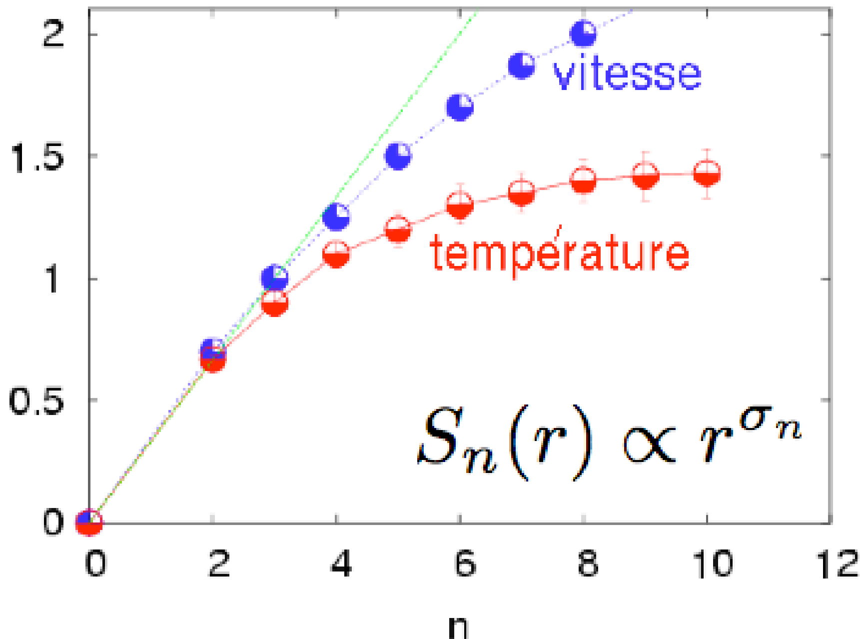
Anomalous Scaling

Is scaling anomalous?

$$S_2(r) = \langle(\delta_r c)^2\rangle \propto r^{2/3}$$
$$(?) \Downarrow (?)$$
$$S_n(r) = \langle(\delta_r c)^n\rangle \propto r^{n/3}$$



Normalized PDF's do not collapse

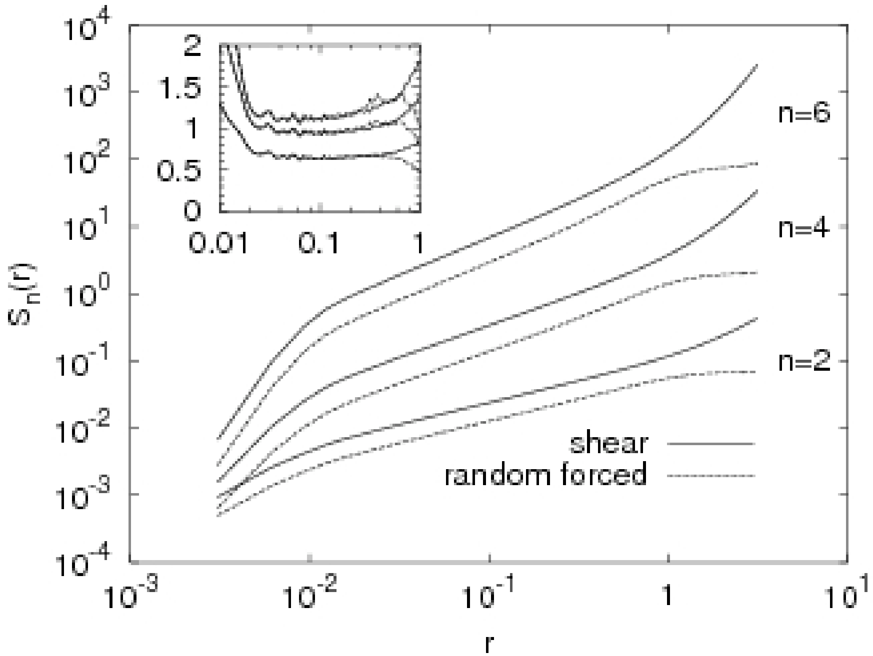


H. Willaime et al. Eur. Phys. J. B 18, 363 (2000)

F. Moisy et al Phys. Rev. Lett. 86, 4827 (2001)

Universality

Is the statistics universal?



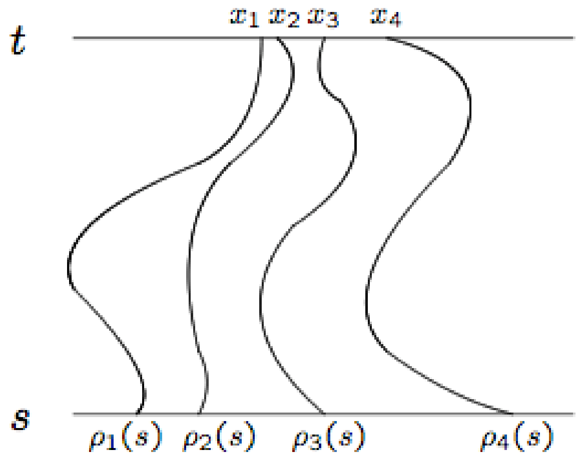
Celani, Lanotte, Mazzino & Vergassola Phys. Rev. Lett. **84**, 2385 (2000)

$$\langle (\delta_r c)^n \rangle = B_n (\epsilon_c^{1/2} \epsilon_v^{-1/6} r^{1/3})^n \left(\frac{L}{r}\right)^{n/3-\sigma_n}$$

Exponents are universal and only depend on the statistics of the velocity

Constants are not universal and depend on the forcing

Zero Modes: the road to anomalous scaling & universality



Anomalous scaling of n-order SF
Can be understood from the
Lagrangian propagator of n-points

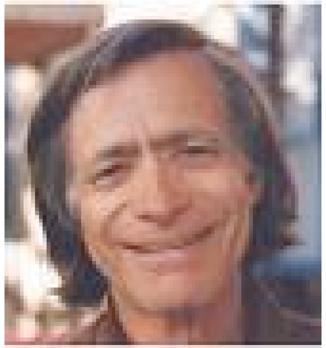
Anomaly comes from ***preserved lagrangian structures***

$$\int \mathcal{P}_n(\underline{y}, s; \underline{x}, t) Z_n(\underline{y}) d\underline{y} = Z_n(\underline{x})$$

D. Bernard *et al*, J. Stat. Phys. **90**, 519 (1998)

Such structures are "written" in the propagator which is universal because does not depend on the forcing

Exactly Soluble model



R.H. Kraichnan

Kraichnan model (1968-1994):

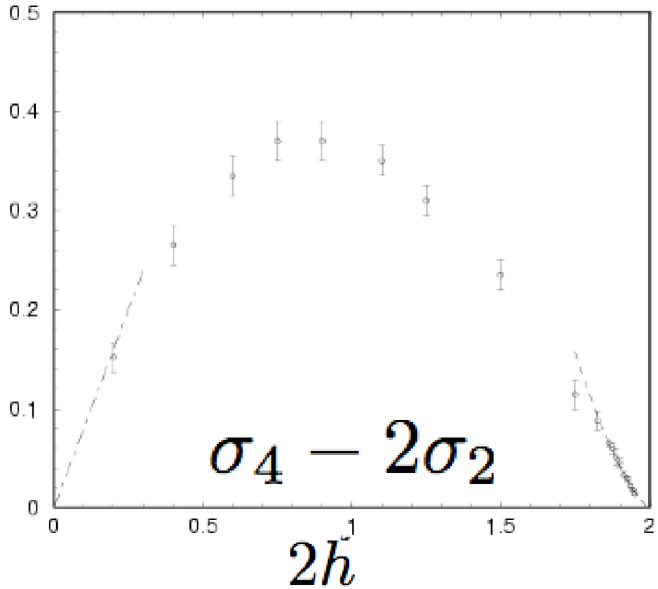
$$\partial_t c + \boldsymbol{v} \cdot \nabla c = \kappa \Delta c + F_c$$

v-is Gaussian & self-similar with

$$\langle \boldsymbol{v}(\boldsymbol{x},t)\rangle = 0$$
$$\langle v_i(\boldsymbol{x},t) v_j(\boldsymbol{y},s)\rangle = \delta(t-s)\left\{D_0\delta_{ij} - \frac{D_1}{2} r^{2h}[(d-1+2h)\delta_{ij} - 2h\hat{r}_i\hat{r}_j]\right\}$$

F_c is also Gaussian and δ-correlated

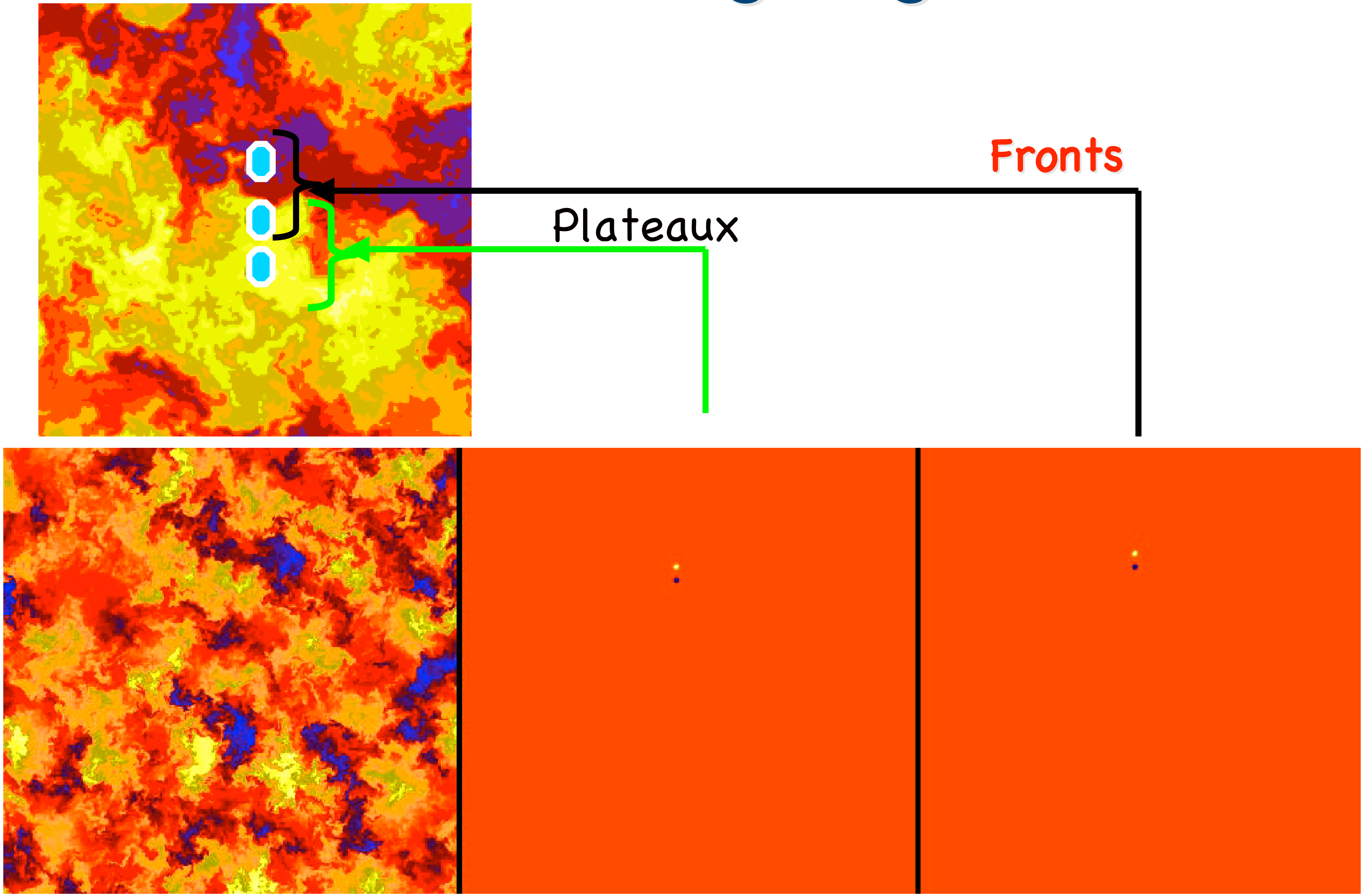
Anomalous scaling can be proved
Numerically and analytically in some limits
Universality of exponents can also be proved



Frisch et al, Phys. Rev. Lett. **80**, 5532 (1998); Phys. Fluids **11**, 2178 (1999)

Gawedzki & Kupiainen, Phys. Rev. Lett. **75**, 3834 (1995)
Chertkov et al Phys. Rev. E **52**, 4924 (1995)
Shraiman & Siggia, C. R. A. S. II **321**, 279 (1995)

Structures and Lagrangian motion



Celani, Cencini & Noullez Physica D **195**, 283 (2004)

Summary of passive scalars

- Passive <==> F_c & velocity are independent
- Lagrangian interpretation <==> Statistics of the scalar determined by the Lagrangian trajectories
- Anomalous scaling hidden in multiparticle trajectories ("*zero modes*"); universality with respect to the forcing: trajectories do not depend on the forcing. Theory for Kraichnan model & verification in realistic turbulent flows (Celani & Vergassola Phys. Rev. Lett. **86**, 424 (2001))
- Review on the current understanding of the problem
  Falkovich et al. Rev. Mod. Phys. **73**, 913 (2001)
  Shraiman & Siggia Nature **405**,639 (2000)

Active scalars

- $$\partial_t a + \boldsymbol{v}\cdot\boldsymbol{\nabla} a = \kappa\Delta a + F_a$$
  $$\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\boldsymbol{\nabla} p + \nu\Delta\boldsymbol{v} + \boxed{\mathcal{F}(a, \boldsymbol{\nabla} a, \ldots)}$$

E.g.

temperature bsq

- $$\partial_t a + \boldsymbol{v}\cdot\boldsymbol{\nabla} a = \kappa\Delta a + F_a$$
  $$v_i(\boldsymbol{x},t) = \int d\boldsymbol{y}\,\Gamma_i[\boldsymbol{x},\boldsymbol{y}]\,a(\boldsymbol{y},t)$$

E.g.

vorticity ns2d

$$F_a \overbrace{\Longrightarrow}^{\mathcal{F}} \boldsymbol{v}$$

Two-way coupling: Lagrangian trajectories are coupled with the scalar forcing

What should we expect on the basis of what we know from passive scalars?

Active Scalars

Two scenarios seem to be possible

[S1] v becomes statistically independent of F_a (at least at small scales) ==>trajectories are "independent" of F_a ==> Universality recovered + passive and active fields in the same flow should share the same statistics

[S2] strong correlation between v and F_a ==> passive and active fields may behave very differently. What about Universality?

A case by case study is needed

Active scalars

We consider 4 examples of passive and active scalars
Evolving in the same flow and compare their statistics

$$\partial_t a + \boldsymbol{v} \cdot \boldsymbol{\nabla} a = \kappa \Delta a + F_a$$

$$\partial_t c + \boldsymbol{v} \cdot \boldsymbol{\nabla} c = \kappa \Delta c + F_c$$

[S2] (mhd) Magnetohydrodynamics (2d)

[S1] (bsq) Thermal convection (2d)

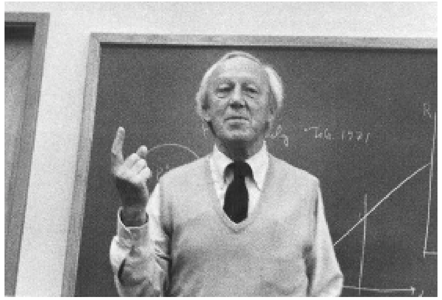
[S1] (eck) Navier-Stokes with Eckman friction (2d)

[S2] (sqe) Surface Quasi-Geostrophic equation (2d)

A. Celani, M. Cencini, A. Mazzino & M. Vergassola, New J. Phys. **6**, 72 (2004)

2d Magnetohydrodynamics

$$\begin{aligned}
\partial_t c + \boldsymbol{v}\cdot\boldsymbol{\nabla} c &= \kappa\Delta c + F_c \\
\partial_t a + \boldsymbol{v}\cdot\boldsymbol{\nabla} a &= \kappa\Delta a + F_a \\
\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla} \boldsymbol{v} &= -\boldsymbol{\nabla} P + \nu\Delta \boldsymbol{v} - \Delta a \boldsymbol{\nabla} a + \cancel{\boldsymbol{f}_{ext}}
\end{aligned}$$



H. Alfvén

Magnetic potential

a

Magnetic field

$\boldsymbol{b} = \boldsymbol{\nabla}^{\perp} a = (\partial_y a, -\partial_x a)$

Lorentz Force

$\mathcal{F} = (\boldsymbol{b}\times\boldsymbol{\nabla})\times\boldsymbol{b} = -\Delta a \boldsymbol{\nabla} a$

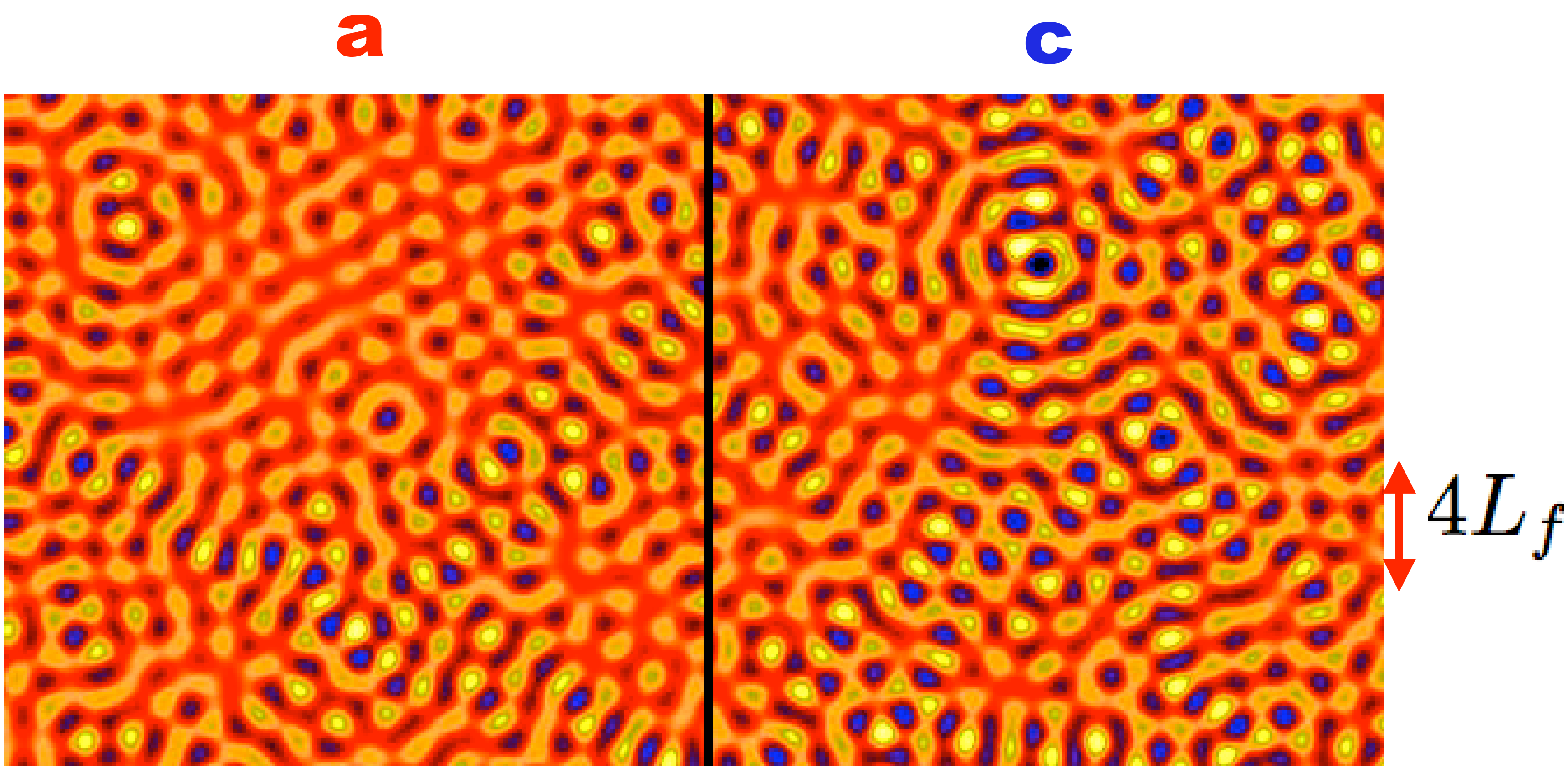
Main conservation laws in the ideal limit

$$A = \int d\boldsymbol{x}\, a^2/2 \quad \text{magnetic pot. energy} \qquad K = \int d\boldsymbol{x}\, \boldsymbol{v}\cdot\boldsymbol{b}/2 \quad \text{cross-helicity}$$

$$E_t = \int d\boldsymbol{x}\, (b^2+v^2)/2 \quad \text{total energy} \qquad E_s = \int d\boldsymbol{x}\, c^2/2 \quad \text{passive scalar energy}$$

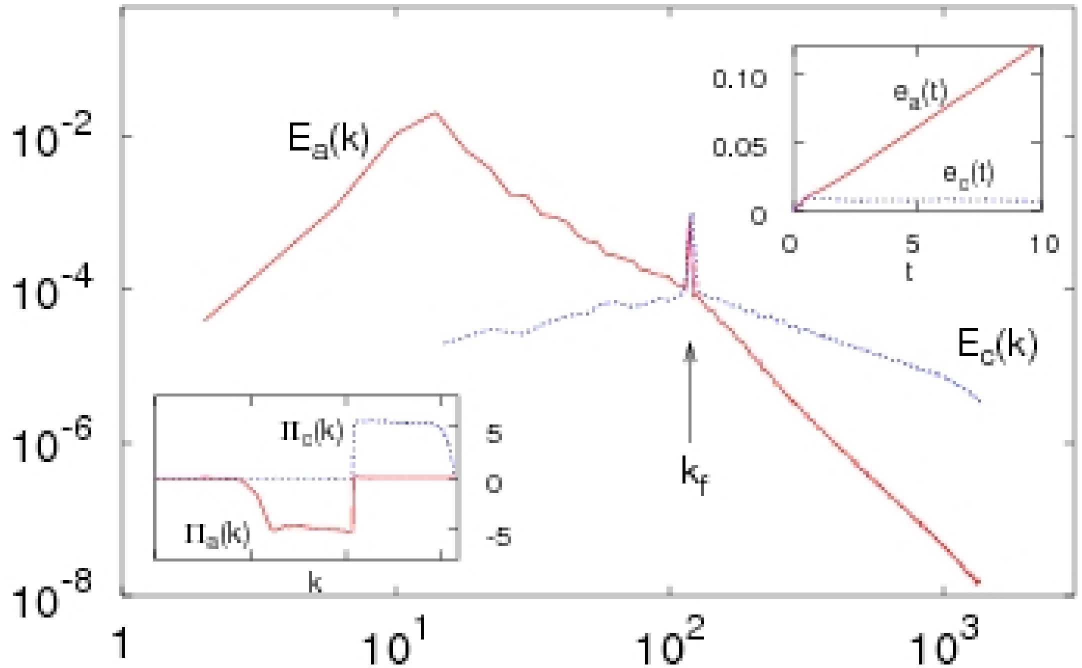
Remember A. Poquet lecture

Dynamics of a and c



F_a & F_c independent realizations of the same Gaussian time uncorrelated, random process acting on the scale L_f

Inverse vs Direct energy cascade



$E_a(k) \propto k^{-7/3}$

Dimensional expectation

Inverse cascade: no dissipative anomaly

$$\epsilon_a = \lim_{\kappa \to 0} \kappa |\nabla a|^2 = 0$$

$$e_a(t) = \tfrac{1}{2}\int a^2(\mathbf{x},t)\,d\mathbf{x} = \tfrac{1}{2}F_0 t$$

non-intermittent for $r>L_f$

Direct cascade: dissipative anomaly

$$\epsilon_c = \lim_{\kappa \to 0} \kappa |\nabla c|^2 \approx \text{input} = \tfrac{1}{2}F_0$$

$$e_c(t) = \tfrac{1}{2}\int c^2(\mathbf{x},t)\,d\mathbf{x} \approx \tfrac{1}{2}F_0 \tau$$

intermittent for $r<L_f$

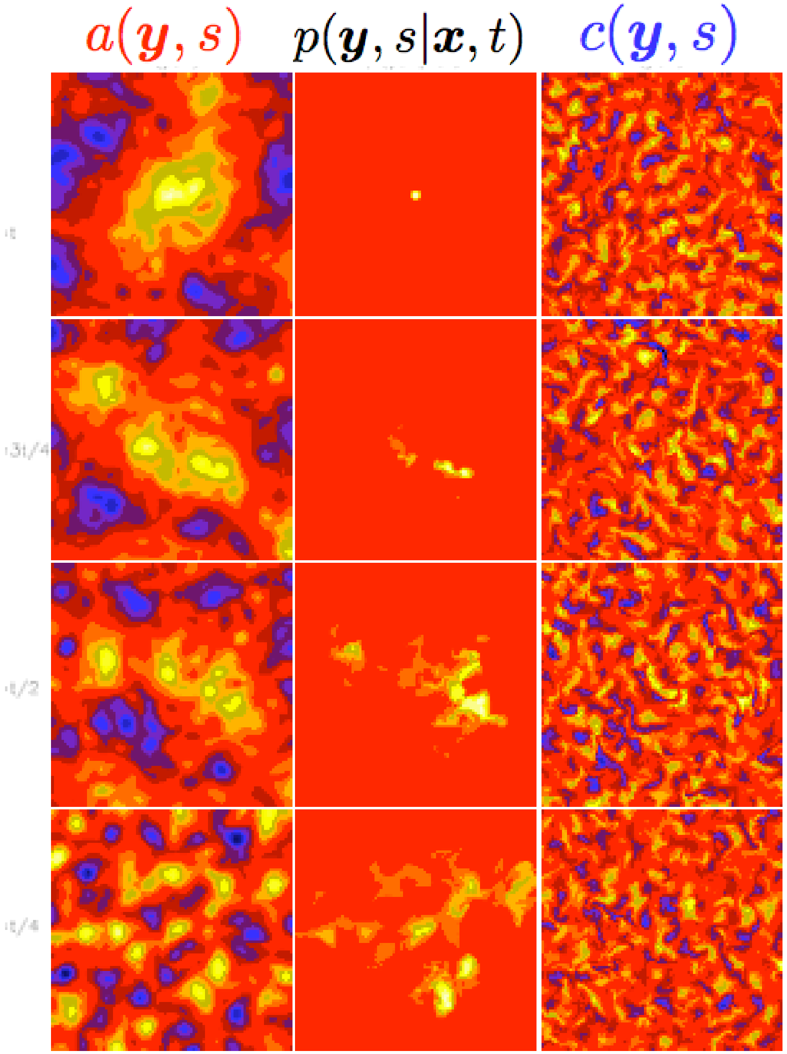
Velocity: is rough both both at $r>L_f$ & $r<L_f$ (away from the dissipative range

==> Lagrangian Paths are not unique! <==

The goal here is to understand the origin of such differences.

Clearly it should be "written" in the correlations between the Lagrangian propagator and the forcing

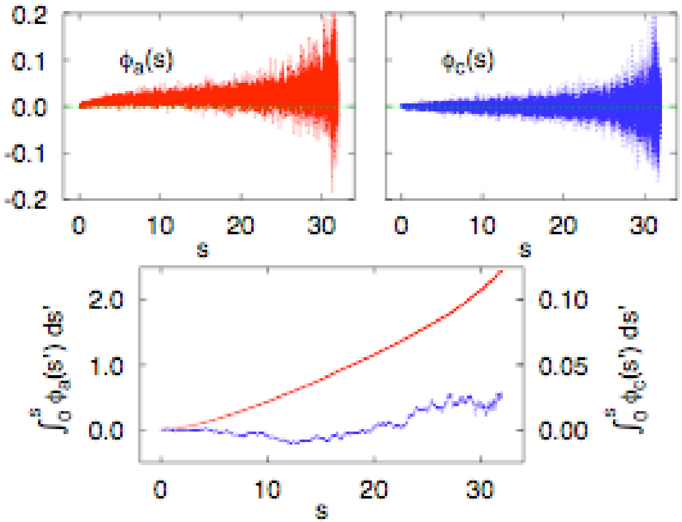
Propagator evolution



$$a(\mathbf{x},t) = \int_0^t ds \, \overbrace{\int p(\mathbf{y},s|\mathbf{x},t)\, F_a(\mathbf{y},s)\, d\mathbf{y}}^{\phi_a(s)}$$

$$c(\mathbf{x},t) = \int_0^t ds \, \overbrace{\int p(\mathbf{y},s|\mathbf{x},t)\, F_c(\mathbf{y},s)\, d\mathbf{y}}^{\phi_c(s)}$$

Correlations between F_a & Lagrangian paths



(A. Celani, M.C., A. Mazzino & M. Vergassola, Phys. Rev. Lett. **89**, 234502 (2002))

Backward Algorithm (A. Celani, M.C., A. Noullez, Physica D **195** 283 (2004))

Lagrangian interpretation of the absence of dissipative anomaly

$$\partial_t a + \boldsymbol{v}\cdot\boldsymbol{\nabla} a = \kappa\Delta a + F_a$$

$$a(\boldsymbol{x},t) = \left\langle \int_0^t ds\, F_a(\boldsymbol{X}(s),s)\right\rangle_X = \int_0^t ds \int F_a(\boldsymbol{y},s)\, p(\boldsymbol{y},s|\boldsymbol{x},t)\, d\boldsymbol{y}$$

$$a^2(\boldsymbol{x},t) = \int_0^t ds \int p(\boldsymbol{y},s|\boldsymbol{x},t)\, F_a(\boldsymbol{y},s)\, d\boldsymbol{y} \int_0^t ds' \int p(\boldsymbol{y}',s'|\boldsymbol{x},t)\, F_a(\boldsymbol{y}',s')\, d\boldsymbol{y}'$$

$$\boxed{a^2(\boldsymbol{x},t) = [\left\langle \int_0^t ds\, F_a(\boldsymbol{X}(s),s)\right\rangle_X]^2}$$

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$$\boxed{a^2(\boldsymbol{x},t) = [\left\langle \int_0^t ds\, F_a(\boldsymbol{X}(s),s)\right\rangle_X]^2}$$

$$\partial_t a^2 + \boldsymbol{v}\cdot\nabla a^2 = \kappa\Delta a^2 + 2aF_a - \boxed{2\epsilon_a} \qquad \epsilon_a \to 0 \qquad \epsilon_a = \kappa|\nabla a|^2$$

Lagrangian interpretation of the absence of dissipative anomaly

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$$a^2(\boldsymbol{x},t) = \int_0^t ds \int p(\boldsymbol{y},s|\boldsymbol{x},t)\; 2\, F_a(\boldsymbol{y},s)\, a(\boldsymbol{y},s) d\boldsymbol{y}$$

Lagrangian interpretation of the absence of dissipative anomaly

$$\partial_t a + \boldsymbol{v}\cdot\boldsymbol{\nabla} a = \kappa\Delta a + F_a$$

$$a(\boldsymbol{x},t) = \left\langle \int_0^t ds\, F_a(\boldsymbol{X}(s),s)\right\rangle_X = \int_0^t ds \int F_a(\boldsymbol{y},s)\, p(\boldsymbol{y},s|\boldsymbol{x},t)\, d\boldsymbol{y}$$

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Lagrangian interpretation of the absence of dissipative anomaly

$$\partial_t a + \boldsymbol{v}\cdot\boldsymbol{\nabla} a = \kappa\Delta a + F_a$$

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$$\partial_t a^2 + \boldsymbol{v}\cdot\nabla a^2 = \kappa\Delta a^2 + 2aF_a - \boxed{2\epsilon_a} \qquad \epsilon_a \to 0 \qquad \epsilon_a = \kappa|\nabla a|^2$$

$$a^2(\boldsymbol{x},t) = 2\int_0^t ds \int d\boldsymbol{y}\, p(\boldsymbol{y},s|\boldsymbol{x},t) F_a(\boldsymbol{y},s) \int_0^s ds' \int d\boldsymbol{y}'\, F_a(\boldsymbol{y}',s') p(\boldsymbol{y}',s'|\boldsymbol{y},s)$$

Lagrangian interpretation of the absence of dissipative anomaly

$$\partial_t a + \boldsymbol{v} \cdot \boldsymbol{\nabla} a = \kappa \Delta a + F_a$$

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$$\boxed{a^2(\boldsymbol{x},t) = \left[\left\langle \int_0^t ds\, F_a(\boldsymbol{X}(s),s) \right\rangle_X\right]^2}$$

$$\partial_t a^2 + \boldsymbol{v} \cdot \nabla a^2 = \kappa \Delta a^2 + 2aF_a - \boxed{2\epsilon_a} \qquad \epsilon_a \to 0$$

$$\epsilon_a = \kappa |\nabla a|^2$$

$$a^2(\boldsymbol{x},t) = 2\int_0^t ds \int d\boldsymbol{y}\, p(\boldsymbol{y},s|\boldsymbol{x},t) F_a(\boldsymbol{y},s) \int_0^s ds' \int d\boldsymbol{y}'\, F_a(\boldsymbol{y}',s') p(\boldsymbol{y}',s'|\boldsymbol{y},s)$$

using $\quad 2\int_0^t ds \int_0^s ds' \Longrightarrow \int_0^t \int_0^t ds ds' \qquad p(\boldsymbol{y}',s'|\boldsymbol{y},s)p(\boldsymbol{y},s|\boldsymbol{x},t) = p(\boldsymbol{y},s;\boldsymbol{y}',s'|\boldsymbol{x},t)$

$$a^2(\boldsymbol{x},t) = \int\int_0^t ds ds' \int\int d\boldsymbol{y} d\boldsymbol{y}'\, p(\boldsymbol{y},s;\boldsymbol{y}',s'|\boldsymbol{x},t) F_a(\boldsymbol{y},s) F_a(\boldsymbol{y}',s') = \langle [\int_0^t ds\, F_a(\boldsymbol{X}(s),s)]^2 \rangle$$

$$\boxed{a^2(\boldsymbol{x},t) = \left\langle \left[\int_0^t ds\, F_a(\boldsymbol{X}(s),s)\right]^2 \right\rangle_X}$$

Lagrangian interpretation of the absence of dissipative anomaly

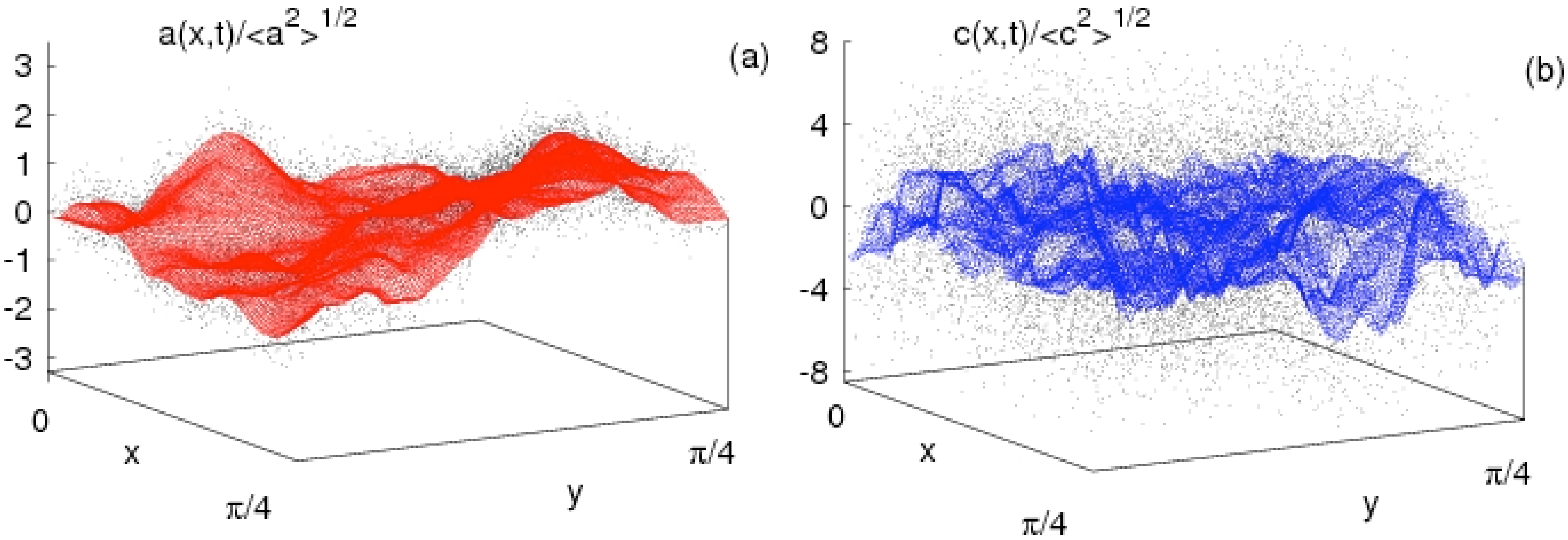
forcing contribution is non-random

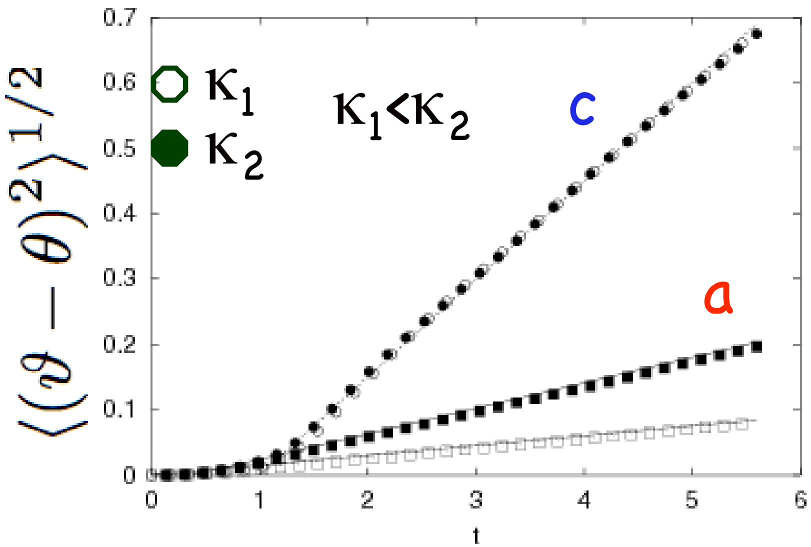
$$\boxed{\left\langle [\int_0^t ds\, F_a(\boldsymbol{X}(s),s)]^n \right\rangle_X = [\left\langle \int_0^t ds\, F_a(\boldsymbol{X}(s),s) \right\rangle_X]^n}$$

Two possibilities

1. Unique Lagrangian path [*collapse onto a unique trajectory*] (compressible flows Gawedzki & Vergassola Physica D **138**, 63 (2000))
2. If many paths, as here, they should organize collectively so that the forcing contribution is the same on all of them.

Presence/Absence of Dissipative anomaly: Lagrangian view

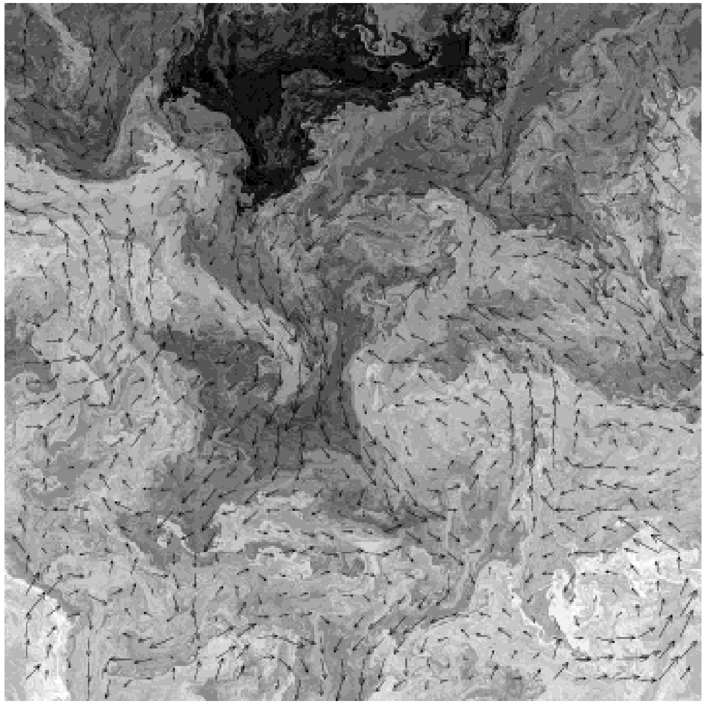




Absence of anomaly:
Particles stick on the surface

Presence of anomaly:
particles diffuse from the surface

2d Turbulent Thermal convection



$$\partial_t a + \boldsymbol{v} \cdot \boldsymbol{\nabla} a = \kappa \Delta a + F_a$$

$$\partial_t c + \boldsymbol{v} \cdot \boldsymbol{\nabla} c = \kappa \Delta c + F_c$$

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = -\boldsymbol{\nabla} p + \nu \Delta \boldsymbol{v} - \boxed{\beta \boldsymbol{g} a} - \alpha \boldsymbol{v}$$

both a & c perform a direct cascade

What about their statistics?

The effect of friction added to stabilize the inverse cascade of velocity

DNS done by Celani, Matsumoto, Mazzino, Vergassola Phys. Rev. Lett. 88, 4503 (2002)

2d Turbulent thermal convection

Phenomenology

Temperature is forced at scale L_f, the bouyancy term forces the velocity inducing an inverse cascade which is arrested by the friction term

Dimensional prediction

$$\delta_r a \sim r^{h_a} \quad \delta_r c \sim r^{h_c} \quad \delta_r v \sim r^{h_v}$$

Yaglom relation

$$\langle(\delta_r \boldsymbol{v}\cdot\hat{\boldsymbol{r}})(\delta_r a)^2\rangle \quad \propto \quad r \qquad \Longrightarrow \qquad h_v + 2h_a = 1$$

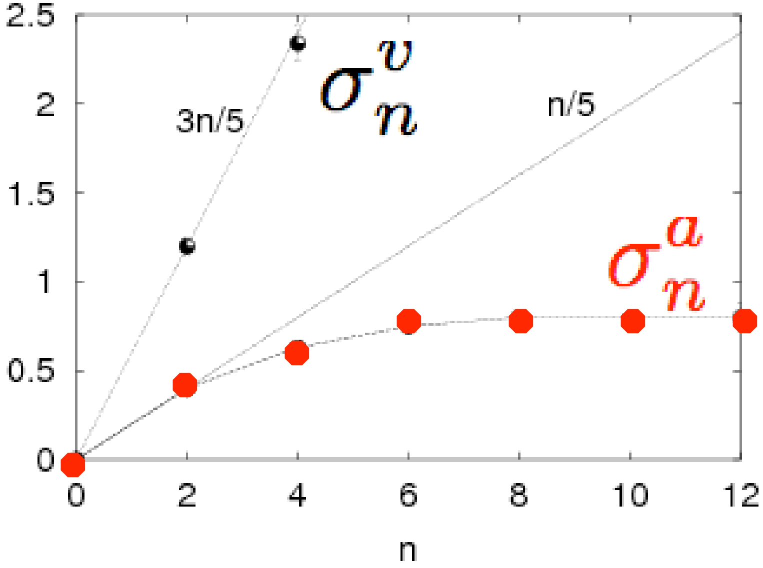
$$\langle(\delta_r \boldsymbol{v}\cdot\hat{\boldsymbol{r}})(\delta_r c)^2\rangle \quad \propto \quad r \qquad \Longrightarrow \qquad h_v + 2h_c = 1$$

balancing $\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}$ & $\beta \boldsymbol{g} a$ $\Longrightarrow$ $2h_v - 1 = h_a$

$$\delta_r a \sim r^{1/5} \quad \delta_r c \sim r^{1/5} \quad \delta_r v \sim r^{3/5}$$

What about high order statistics?

2d turbulent thermal convection

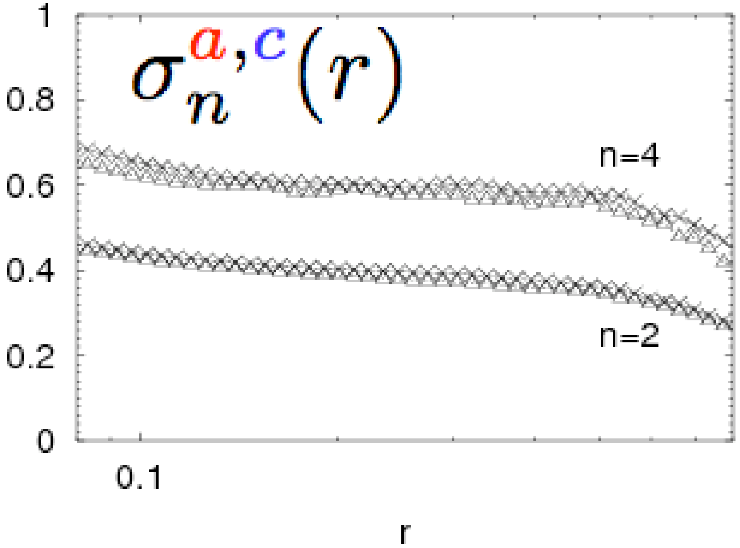


$$S_n^a(r) = \langle (\delta_r a)^n \rangle \propto r^{\sigma_n^a} \qquad \sigma_n^a \neq n/5$$

$$S_n^v(r) = \langle (\delta_r v)^n \rangle \propto r^{\sigma_n^v} \qquad \sigma_n^v = 3n/5$$

Self-Similar Velocity field

Anomalous scaling for the temperature

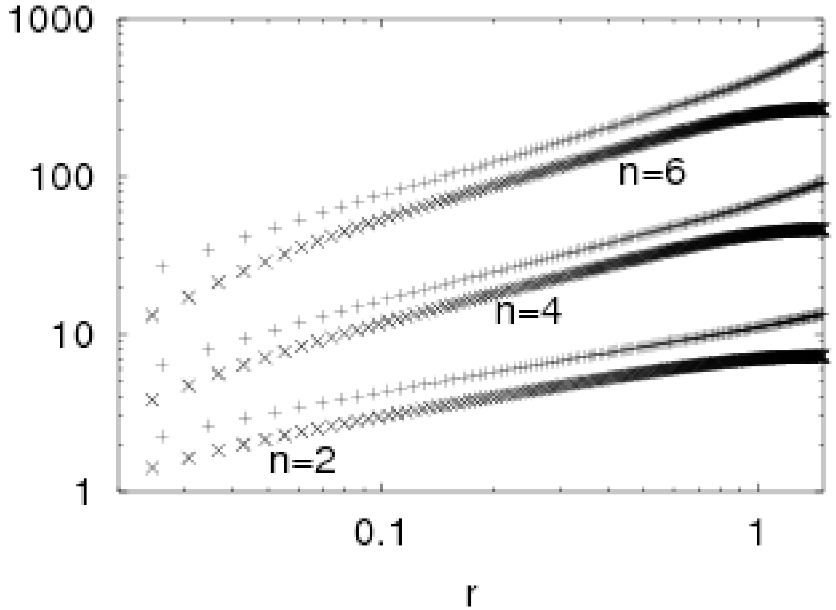


Active and Passive fields have the same exponents

$$\sigma_n^c = \sigma_n^a$$

Universality?

Universality in 2d thermal convection



$$\langle F_a(\boldsymbol{x},t)F_a(\boldsymbol{y},s)\rangle = \Phi(r)\delta(t-s) \quad \text{random forcing}$$

$$F_a(\boldsymbol{x},t) = \gamma \boldsymbol{g}\cdot\boldsymbol{v} \quad \text{imposed gradient}$$

Universality with respect to forcing

2d turbulent convection fits

in the passive scalar scenario

Celani et al Phys. Rev. Lett. 88, 4503 (2002) proposed the following interpretation

Velocity performs an inverse cascade. Inverse cascades are known to be non-intermittent and universal

$\Downarrow$

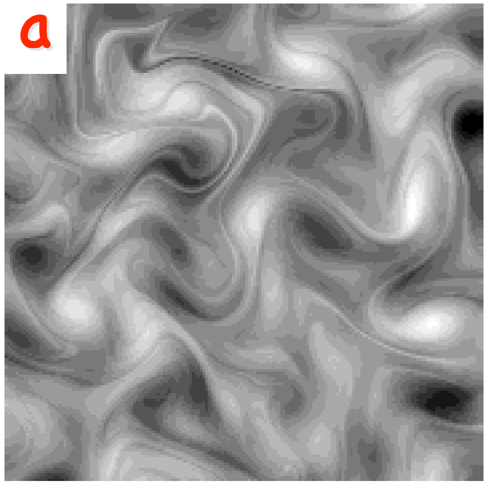
Lagrangian statistics (determined by the velocity) are universal and so the advected active scalar field

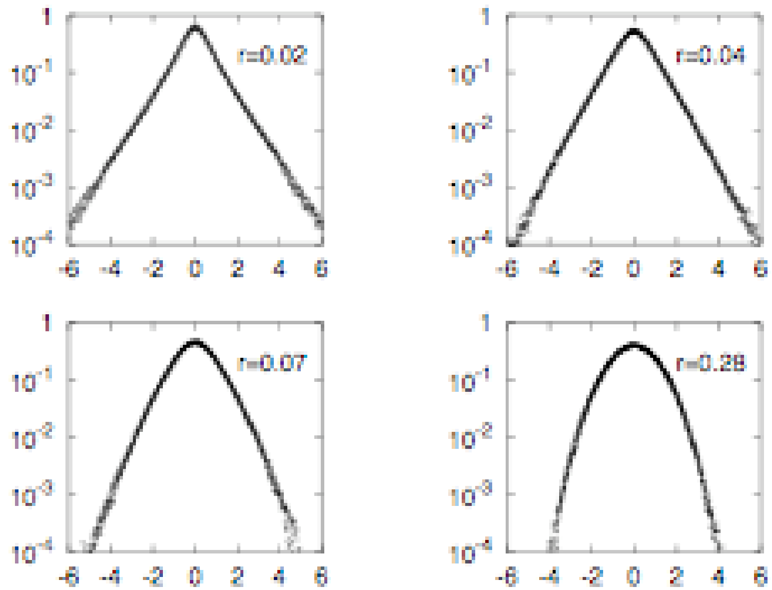
2d Eckmann turbulence

$$\partial_t a + \boldsymbol{v}\cdot\boldsymbol{\nabla} a = \kappa\Delta a - \boxed{\alpha a} + F_a$$ a vorticity with eckman friction

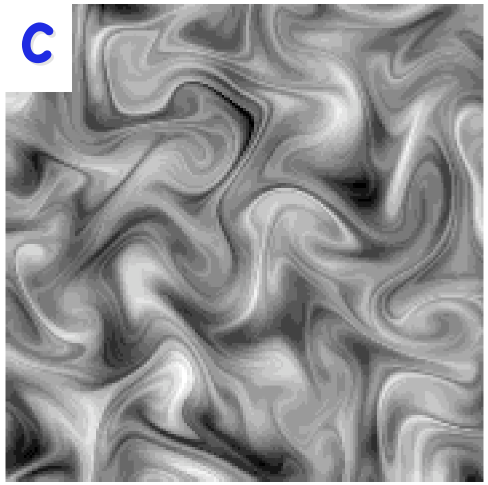
$$\partial_t c + \boldsymbol{v}\cdot\boldsymbol{\nabla} c = \kappa\Delta c - \boxed{\alpha c} + F_c$$ c decaying scalar

a





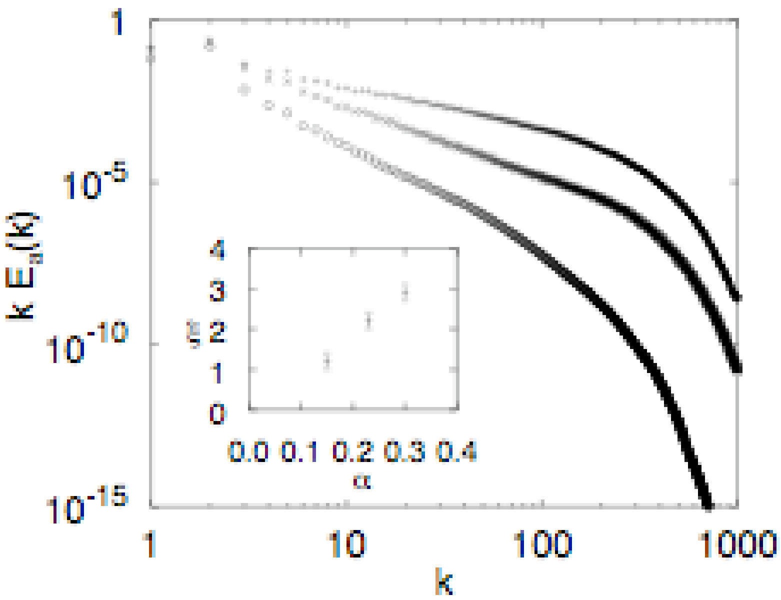
c



$$P(\delta_r a) = P(\delta_r c)$$

(Boffetta et al. Phys. Rev. E **66**, 026304 (2002))

2d Eckmann turbulence



The scalars are force at large scale and the velocity is smooth with spectrum

$$E_v(k) \approx k^{-3-\xi(\alpha)}$$

Which depends on the friction coefficent

(Boffetta et al. Phys. Rev. E **66**, 026304 (2002);Europhys. Lett. 71 590 (2005) - Nam, et al Phys. Rev. Lett. **84**, 5134 (2000); Phys. Rev E **71** 066313 (2005))

Smooth velocity=> uniqueness of the Lagrangian paths plus the regularization induced by the friction are the ingredients to explain the behavior of a & c

Warning: universality may be lost

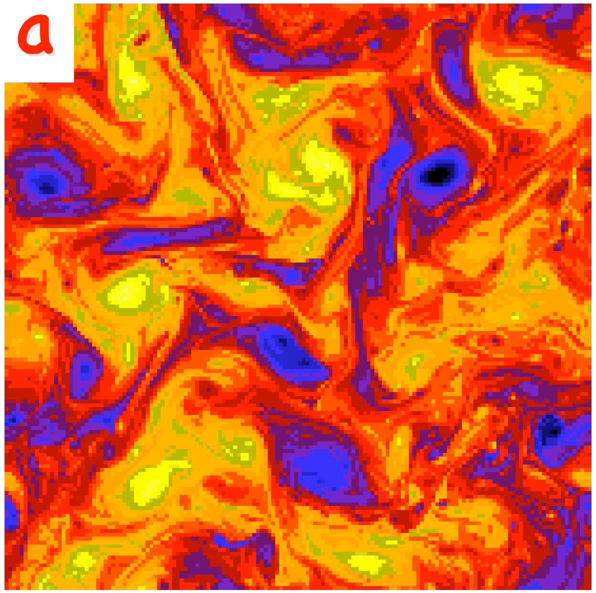
Surface Quasi Geostrophic turbulence

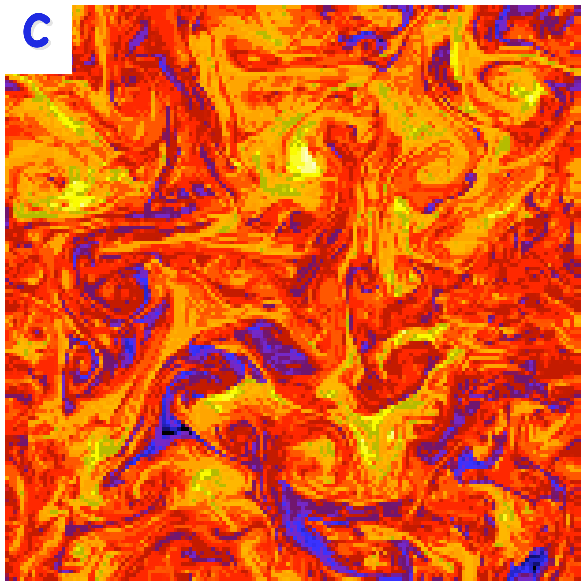
$$\partial_t a + \boldsymbol{v} \cdot \boldsymbol{\nabla} a = \kappa \Delta a + F_a \qquad \partial_t c + \boldsymbol{v} \cdot \boldsymbol{\nabla} c = \kappa \Delta c + F_c$$

$$v_i(\boldsymbol{x},t) = \int d\boldsymbol{y}\, \epsilon_{ij} \partial_{x_i} |\boldsymbol{x} - \boldsymbol{y}|^{-1}\, a(\boldsymbol{y},t)$$

a –fluid density on the flat surface of an infinite high fluid column

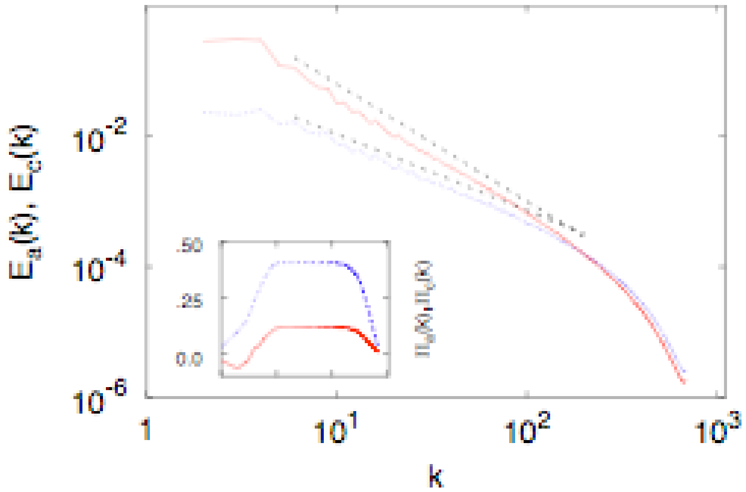
Pierrehumbert et al. Chaos Sol. Fract. **4**, 1111 (1994); J.Fluid Mech. **282**, 1 (1995)





Celani, Cencini, Mazzino & Vergassola, New J. Phys. **6**, 37 (2004)

Surface Quasi Geostrophic turbulence

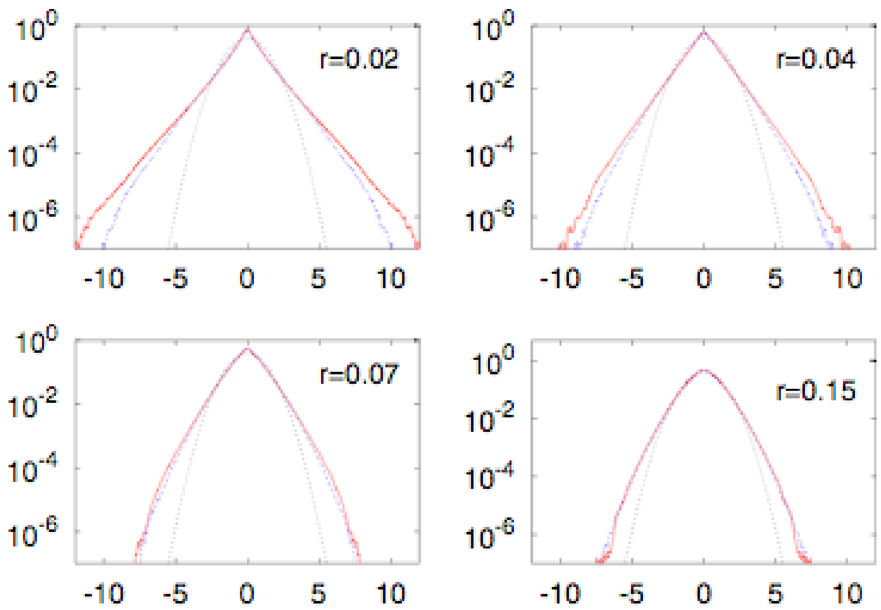


Dimensional Prediction

$$E_a(k) \sim E_c(k) \sim k^{-5/3}$$

Observed

$$E_a(k) \sim k^{-1.8} \qquad E_c(k) \sim k^{-1.17}$$



PDFs are different

Summary

mhd A/P Inverse/direct cascade, absence/presence of anomaly. Large scales not-intermittent and universal! Small scales universality?

bsq A/P Direct cascade with dissipative anomaly. Same statistics and universality

eck A/P absence of anomaly, intermittency with same exponents. Universality may be lost!

sqe A/P Direct cascade with anomaly. the statistics is different! Universality?

Take home messages

Euler

Lagrange

Eulerian and Lagrangian approaches are complementary

The Lagrangian provides the possibility of clear physical interpretations

Universality in active scalars is an open issue

Perspectives

Back to NS turbulence

- NS-Turbulence is a problem of active vector transport
  (P. Constantin, Commun. Math. Phys., **216** 663 (2001))
- Studying co-evolving active and passive vector
  (R. Benzi et al. Eur. Phys. J. B, **24**, 125 (2001))

Mhd-turbulence (Univerality?)

- Iroshnikov-Kraichnan scaling: $E_a(k) \sim k^{-7/2}$
  D. Biskamp & E. Schwarz, Phys. Plasmas **8**, 3282 (2001)
- Kolmogorov scaling: $E_a(k) \sim k^{-11/3}$
  H. Politano et al., Europhys. Lett. **43**, 516 (1998) M. K. Verma et al., J. Geophys. Res. **101**, 21619 (1996)

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A. Celani, T. Matsumoto, A. Mazzino & M. Vergassola *Phys. Rev. Lett.* **88**, 054503 (2002)

E.S.C. Ching, Y. Cohen, T. Gilbert & I. Procaccia, *Europhys. Lett.* **60**, 369 (2003)
Phys. Rev. E 67, 016304 (2003)

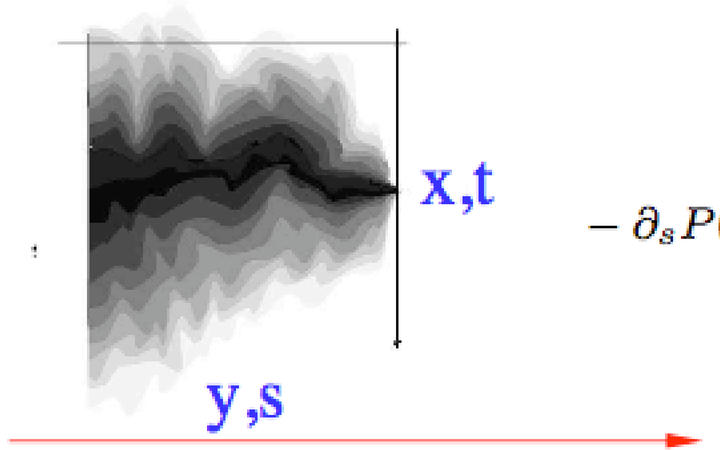
A. Celani, M. Cencini, A. Mazzino & M. Vergassola, *Phys. Rev. Lett.* **89**, 234502 (2002)
(free) http://www.iop.org/EJ/abstract/1367-2630/6/1/072 New J. Phys. **6**, 37 (2004)

G. Boffetta, A. Celani, S. Musacchio & M. Vergassola, Phys. Rev. E **66**, 026304 (2003)

Extras

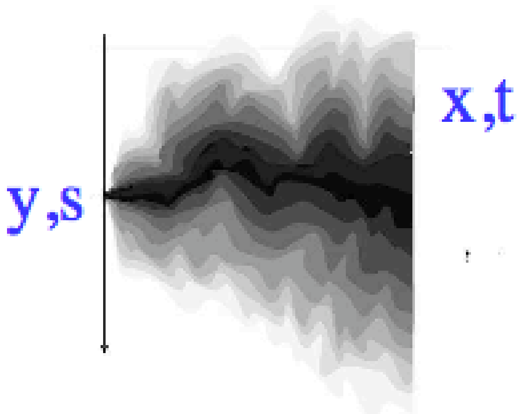
Lagrangian propagator

Backward propagator



$$-\partial_s P(\boldsymbol{y},s|\boldsymbol{x},t) - \boldsymbol{v}(\boldsymbol{y},s)\cdot\boldsymbol{\nabla}_y P(\boldsymbol{y},s|\boldsymbol{x},t) = \kappa\Delta_y P(\boldsymbol{y},s|\boldsymbol{x},t)$$

$$P(\boldsymbol{y},s=t|\boldsymbol{x},t) \quad = \quad \delta(\boldsymbol{x}-\boldsymbol{y})$$



$$\partial_t P(\boldsymbol{y},s|\boldsymbol{x},t) + \boldsymbol{\nabla}_x\cdot[\boldsymbol{v}(\boldsymbol{x},t)P(\boldsymbol{y},s|\boldsymbol{x},t)] \;=\; \kappa\Delta_x P(\boldsymbol{y},s|\boldsymbol{x},t)$$

Forwad propagator

Kraichnan model & Zero Modes

If v is Gaussian, self-similar ($\delta_r v \sim r^h$) & δ-correlated in time, closed equation for the correlation function

$$\mathcal{C}_N(\underline{\mathbf{x}}) \equiv \langle c(\mathbf{x}_1, t) \dots c(\mathbf{x}_N, t) \rangle \qquad \mathcal{C}_N(\lambda \underline{\mathbf{x}}) = \lambda^{\sigma_N} \mathcal{C}_N(\underline{\mathbf{x}})$$

$$\boxed{\partial_t \mathcal{C}_N + \mathcal{M}_N \mathcal{C}_N = \mathcal{F} \otimes \mathcal{C}_{N-2}}$$

$\mathcal{M}_N$ Linear Op. $\Longrightarrow$ $\qquad \mathcal{C}_N = \mathcal{Z}_N + \mathcal{I}_N \qquad \mathcal{M}_N \mathcal{Z}_N = 0$

$$\mathcal{I}_N(\lambda \underline{\mathbf{x}}) = \lambda^{\sigma_N^{dim}} \mathcal{I}_N(\underline{\mathbf{x}}) \qquad \mathcal{Z}_N(\lambda \underline{\mathbf{x}}) = \lambda^{\sigma_N} \mathcal{Z}_N(\underline{\mathbf{x}})$$

$\mathcal{Z}_N$ does not depend on $F_c \Longrightarrow$ Universality

K. Gawędzki & A. Kupiainen, Phys. Rev. Lett. **75**, 3834 (1995) M. Chertkov *et al* Phys. Rev. E **52**, 4924 (1995) B.I. Shraiman & E.D. Siggia, C. R. A. S. II **321**, 279 (1995)