Small-Scale Anisotropy in High Reynolds Number Turbulence

-- Universality of the 2nd Kind --

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Turbulence
= a system of huge degree of freedom

A paradigm of
Studies of systems of huge degree of freedom

→

Thermodynamics and Statistical Mechanics
for thermal equilibrium state
I. **Two kinds of universality**

characterizing the macroscopic state of the equilibrium system

not only

a) Equilibrium state itself, like Boyle-Charles’ law

but also

b) **Response** to disturbance

↔ **Universality of the 2nd Kind**

II. Thermal equilibrium state ↔ Universal equilibrium state at small scale

influence of external force, mean flow, etc. at small scale

may be regarded as disturbance.

**How does the equilibrium state respond to the disturbance?**
Equilibrium state
at thermal equilibrium state
disturbance
response
1905, Einstein, $D = \mu kT$,
the first example of FD-elation → Perrin’s experiment.

1928, Nyquist’s theorem on thermal noise:
$$P(f) = 4kT \ Re(Z(f))$$

1931, Onsager’s reciprocal theorem:
$$J = CX, \quad C = T^T C$$

generalized flux, generalized force
\[ J = C X \]

Generalized Flux vs. Generalized Force

e.g.
- Density Flux vs. Density gradient \( J = C \text{ grad } \rho \)
- Heat Flux vs. Temperature gradient \( J = C \text{ grad } T \)
- Electric Current vs. External electric field \( J = \sigma \ E = \sigma \text{ grad } \phi \),
  \( l=E/R \), Ohm’s law

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Momentum Flux vs. Strain rate

\[ \tau_{ij} = C_{ijmn} S_{mn}, \]

(Newton’s law)

coupling only between tensors of the same order.
Thermal Equilibrium system

Disturbance

$\mathbf{X} \quad \nabla \phi \quad \nabla T \quad \nabla c$

Equilibrium state

Linear response

$F = CX$ \hspace{1cm} (Hooke’s law)

$J = C' \nabla \phi = \sigma E$ \hspace{1cm} (Ohm’s law)

$J = C'' \nabla T$ \hspace{1cm} (Fourier’s law)

$J = C''' \nabla c$ \hspace{1cm} (Fick’s law)
History:
Universality in Response

to disturbances, near equilibrium state

1905, Einstein, $D=\mu kT$,
the first example of FD-elation→ Perrin’s experiment.

1928, Nyquist’s theorem on thermal noise:
$$P(f)=4kT \Re(Z(f))$$

1931, Onsager’s reciprocal theorem:
$$J = CX, \quad C^T C$$

  generalized flux,  generalized force

1950-60, Nakano, Kubo
Linear Response Theory
Equilibrium State of Turbulence
Turbulence ?

Equilibrium state

disturbance

response
DNS

Energy Spectrum a la Kolmogorov (K41)

From Kaneda & Ishihara (2006)
Equilibrium state

a la Kolmogorov K41

response

disturbance
Disturbance

1. Mean Shear
2. Buoyancy by Stratification
3. Magneto-Hydrodynamic Force
I. Mean Shear

See Ishihara et al. (2002)
Turbulent Shear Flow

NS-equation \hspace{1cm} \text{Local co-ordinate} \hspace{1cm} \mathbf{v} = \mathbf{v}_m \mathbf{V} = \mathbf{V}_m + \mathbf{v}_T

\[
\frac{\partial \mathbf{v}}{\partial t} = - (\mathbf{v} \cdot \nabla) \mathbf{v} - \nabla q + \nu \nabla^2 \mathbf{v} + \mathbf{M},
\]

\[
M_i = S_{mn} r_n \frac{\partial \mathbf{V}_m}{\partial r_n} + S_{ij} \mathbf{v}_j
\]

Effect of mean shear \hspace{1cm} \text{Local strain rate of mean flow}

for \( r \ll L, \mathbf{v} \sim S r \)

\[
(\mathbf{v} \cdot \nabla) \mathbf{v} \sim \mathbf{v}_T^2 / \ell, \hspace{1cm} \nu \nabla^2 \mathbf{v} \sim \nu \mathbf{v}_T / \ell^2, \hspace{1cm} M \sim S \mathbf{v}_T,
\]

\[
\tau_N \sim \ell / \mathbf{v}_T, \hspace{1cm} \tau_v \sim \ell^2 / \nu, \hspace{1cm} \tau_E \sim 1 / S,
\]

\[
\frac{M}{(\mathbf{v} \cdot \nabla) \mathbf{v}} \sim \frac{S \mathbf{v}_T}{\mathbf{v}_T^2 / \ell} = \frac{S \ell}{\mathbf{v}_T} \propto \frac{S \ell^{2/3}}{\epsilon^{1/3}} \ll 1
\]

for \( \ell \ll \ell_E = (\epsilon^{1/3} / S)^{3/2} \).
Effect of Mean Flow

Homogenous Mean Shear Flow:

\[ U_i = S_{ij} x_j \]

- in the inertial subrange of homogeneous turbulent shear flow;

\[ Q_{ij}(k) = \left\langle u_i(k)u_j(-k) \right\rangle = ? \]

Let \( \delta(k) = \tau_k / T \), (where \( \tau_k = 1/ (\varepsilon^{1/3} k^{2/3}) \), \( T = 1/S \))

\[ \delta(k) \equiv S/[kv(k)] \sim S/(k^{2/3} \varepsilon^{1/3}) \quad k = 1/l \]

Assume (1) \( \delta \ll 1 \), for large enough \( k \)
(2) expand in powers of \( \delta \)
Two kind of measures characterizing the universal equilibrium state

\[ <u_i(k)u_j(-k)> = Q_{ij}(k) = Q_{ij}^0(k) + C_{ij\alpha\beta}(k)S_{\alpha\beta} \]

1) Equilibrium state itself

Not only

but also

2) Response to disturbance

Similar to stress vs. rate of strain relation:

\[ \tau_{ij} = C_{ij\alpha\beta}S_{\alpha\beta} \]

(justified by the Linear response theory for non-equilibrium system)
Anisotropic part

at small scale of turbulent shear flow

\begin{align*}
\langle u_i(k)u_j(-k) \rangle &= Q_{ij}(k) = Q_{ij}^0(k) + C_{ij\alpha\beta}(k)S_{\alpha\beta} \\
C_{ij\alpha\beta}(k) &= a(k) \left[ P_{i\alpha}(k)P_{j\beta}(k) + P_{i\beta}(k)P_{j\alpha}(k) \right] + b(k)P_{ij}(k)\hat{k}_\alpha\hat{k}_\beta \\
a(k) &= A \varepsilon^{1/3} k^{-13/3}, \quad b(k) = B \varepsilon^{1/3} k^{-13/3}
\end{align*}

cf. Lumley (1967)  
Cambon & Rubinstein (2006)

Only 2 (universal) parameters, A and B

Is this correct? What are the values of A and B?
Method of numerical experiment

DNS of homogeneous turbulence with the simple shear flow

\[ U = \begin{pmatrix} Sx_2 \\ 0 \\ 0 \end{pmatrix} \]

Initial condition: isotropic turbulence

Resolution: \( 512^3 \)

\( k_{\text{max}} \eta = 1 \)

\( S = 0.5, 1.0 \)

Observe \( E_{ij}(k) = \sum_{p=k} Q_{ij}(p) \), \( E_{ij}^{ab}(k) = \sum_{p=k} \hat{p}_a \hat{p}_b Q_{ij}(p) \)

especially,

\( E_{12}(k), E_{ii}^{12}(k), E_{11}^{12}(k), E_{22}^{12}(k), E_{33}^{12}(k), E_{12}^{11}(k), E_{12}^{22}(k), E_{12}^{33}(k) \)

\[ E_{12}(k) = \frac{4\pi}{15} (7A-B) \xi, E_{12}(k), \quad \frac{1}{2} E_{ii}^{12}(k) = \frac{4\pi}{15} (-A+B) \xi, L \]

\( \xi = \varepsilon^{1/3} k^{-7/3} S \)

Estimate A and B, and check consistency.
Anisotropic Energy spectrum of homogeneous turbulent shear flow

\[
E_{ij}(k) = \sum_{p=k} \langle u_i(p)u_j(-p) \rangle
\]

\[
E_{ij}^{ab}(k) = \sum_{p=k} \hat{p}_a \hat{p}_b \langle u_i(p)u_j(-p) \rangle
\]

Theoretical predictions:

\[
\frac{1}{2} E_{ii}^{12}(k) = \frac{4\pi}{15} (-A + B) \xi
\]

\[
E_{12}(k) = \frac{4\pi}{15} (7A - B) \xi
\]

\[
E_{12}^{11}(k) = E_{12}^{22}(k) = \frac{4\pi}{105} (13A - 3B) \xi
\]

\[
E_{12}^{33}(k) = \frac{4\pi}{105} (23A - B) \xi, L
\]

where \( \xi = S \varepsilon^{1/3} k^{-7/3} \).
Consistency

\[ -E_{12}^{11}(k), -E_{12}^{22}(k), -E_{12}^{33}(k) \]

\[ -E_{11}^{12}(k), -E_{22}^{12}(k), -E_{33}^{12}(k) \]

Theoretical predictions:

\[ E_{12}^{11}(k) = E_{12}^{22}(k) = \frac{4\pi}{105} (13A - 3B)\xi \]

\[ E_{12}^{33}(k) = \frac{4\pi}{105} (23A - B)\xi \]

\[ E_{11}^{12}(k) = E_{22}^{12}(k) = \frac{16\pi}{105} (-2A + B)\xi \]

\[ E_{33}^{12}(k) = \frac{8\pi}{105} (A + 3B)\xi \]

See also Yoshida et al. (2003)
II. Stratification

See Kaneda & Yoshida (2004)
Boussinesq approximation

\[ \frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p + \nu \nabla^2 \mathbf{u} - N \rho e_3, \]

\[ \nabla \cdot \mathbf{u} = 0, \]

\[ \frac{\partial \rho}{\partial t} = -(\mathbf{u} \cdot \nabla)\rho + \kappa \nabla^2 \rho + Nu_3, \]

\[ \tau_N \sim \ell / \nu, \quad \tau_v \sim \ell^2 / \nu, \quad \tau_E \sim 1 / N, \]

\[ \tau_N / \tau_E \sim \delta(k) \equiv N/[kv(k)] \ll 1, \]

\[ B_z(k, t) = -\frac{1}{(2\pi)^3} \int d^3r \langle u_z(x+r, t)\rho(x, t) \rangle e^{-i\mathbf{k} \cdot \mathbf{r}}. \]

\[ P(k, t) = \frac{1}{(2\pi)^3} \int d^3r \langle \rho(x+r, t)\rho(x, t) \rangle e^{-i\mathbf{k} \cdot \mathbf{r}}, \]
\[
\begin{align*}
Q_{ij}(k) &= Q_{ij}^{(0)}(k) + \Delta Q_{ij}(k), & \Delta Q_{ij}(k) &= Q_{ijm}^{(1)}(k)N_m + Q_{ijmn}^{(2)}(k)N_mN_n + ..., \\
P(k) &= P^{(0)}(k) + \Delta P(k), & \Delta P(k) &= P_m^{(1)}(k)N_m + P_{mn}^{(2)}(k)N_mN_n + ..., \\
B_i(k) &= B_i^{(0)}(k) + \Delta B_i(k), & \Delta B_i(k) &= B_{im}^{(1)}(k)N_m + B_{imn}^{(2)}(k)N_mN_n + ..., \\
N &= N e_3
\end{align*}
\]

\[
\begin{align*}
\Delta Q_{ij}(k) &= \left[ q_1(k)P_{i3}(k)P_{j3}(k) + q_2(k)P_{ij}(k)\frac{k_3^2}{k^2} + q_3(k)P_{ij}(k) \right]N^2, \\
\Delta P(k) &= \left[ p_1(k)\frac{k_3^2}{k^2} + p_2(k) \right]N^2, \\
\Delta B_i(k) &= b(k)P_{i3}(k)N,
\end{align*}
\]

Kolmogorov scaling:

\[
q_i(k) = \alpha_i k^{-5}, \quad p_j(k) = \beta_j k^{-5}, \quad b(k) = \gamma e^{1/3} k^{-13/3},
\]
From Kaneda & Yoshida (2004)
Angular dependence

From Kaneda & Yoshida (2004)
From Kaneda & Yoshida (2004)
III. MHD

See Ishida & Kaneda (2007)
MHD approximation

\[
\frac{\partial u}{\partial t} + (u \cdot \text{grad}) u = -\frac{1}{\rho} \text{grad} p + \nu \Delta u + \frac{1}{\rho} F, \\
\text{div} u = 0,
\]

\[
\frac{\partial B}{\partial t} = (B \cdot \text{grad}) u - (u \cdot \text{grad}) B + \eta_e \Delta B
\]

\[
\text{div} B = 0
\]

\[
F = \frac{1}{\mu_e} (B \cdot \text{grad}) B
\]

\[
F' = \frac{1}{\mu_e} (B \cdot \text{grad}) B
\]

\[
\approx \frac{1}{\mu_e} (B_0 \cdot \text{grad}) b
\]

\[
= -\frac{\sigma_e}{\rho_e} \Delta^{-1} (B_0 \cdot \text{grad})^2 u + O(b^2)
\]
\[(\mathbf{v} \cdot \text{grad})\mathbf{v} \sim \frac{v_{\ell}^2}{\ell}, \quad \nu \nabla \mathbf{v} \sim \frac{\nu v_{\ell}}{\ell^2}, \quad F \sim M v_{\ell}, \quad \left( M \equiv \frac{\sigma_e B_0^2}{\rho} \right) \]

\[
\frac{|F|}{|(\mathbf{v} \cdot \text{grad})\mathbf{v}|} \sim \delta_f = \frac{M \ell}{v_{\ell}},
\]

\[|F| \ll |(\mathbf{v} \cdot \text{grad})\mathbf{v}| \quad \text{for} \quad \delta_f \ll 1.\]
Turbulent Shear Flow

NS-equation Local co-ordinate \( \mathbf{v} = \langle \mathbf{v} \rangle + \tilde{\mathbf{v}} \)

\[
\frac{\partial}{\partial t} \tilde{\mathbf{v}}(r, t) = - (\tilde{\mathbf{v}} \cdot \nabla) \tilde{\mathbf{v}} - \nabla q + \nu \nabla^2 \tilde{\mathbf{v}} + \mathbf{M},
\]

\[M_i = S_{mn} r_n \frac{\partial \tilde{v}_j}{\partial r_m} + S_{ij} \tilde{v}_j\]

For \( r \ll L, \langle \mathbf{v} \rangle \sim S r \)

\[(\tilde{\mathbf{v}} \cdot \nabla) \tilde{\mathbf{v}} \sim v_{\ell}^2 / \ell, \quad \nu \nabla^2 \tilde{\mathbf{v}} \sim \nu v_{\ell} / \ell^2, \quad \mathbf{M} \sim S v_{\ell},\]

\[\tau_N \sim \ell / v_{\ell}, \quad \tau_v \sim \ell^2 / \nu, \quad \tau_E \sim 1 / S,\]

\[
\frac{\mathbf{M}}{(\tilde{\mathbf{v}} \cdot \nabla) \tilde{\mathbf{v}}} \sim \frac{S v_{\ell}}{v_{\ell}^2 / \ell} = \frac{S \ell}{v_{\ell}} \propto \frac{S \ell^{2/3}}{\epsilon^{1/3}} \ll 1
\]

For \( \ell \ll \ell_E = (\epsilon^{1/3} / S)^{3/2} \).
Disturbance : $X$

External force due to the magnetic field

Isotropic turbulence
( Equilibrium state )

Response; Anisotropic

$C \times X$

Approximately linear in “X”

$$Q_{ij}(k) = Q_{ij}^{(0)}(k) + \Delta Q_{ij}(k),$$

$$\Delta Q_{ij}(k) = C_{ijmn}(k) \alpha_{mn},$$

$$\alpha_{mn} = \frac{\sigma_e B_{0m} B_{0n}}{\rho}$$
response of isotropic turbulence

\[ \Delta Q_{ij}(k) = C_{ijmn}(k) \alpha_{mn} \]

\[ = \left[ \frac{1}{2} q_1(k) (P_{im}(k)P_{jn}(k) + P_{jm}(k)P_{in}(k)) \right] + q_2(k)P_{ij}(k) \frac{k_mk_n}{k^2} + q_3(k)P_{ij}(k)\delta_{mn} \alpha_{mn} \]

3 (universal) parameters in the inertial subrange
anisotropic parts

• To detect $q_1(k)$ and $q_2(k)$, define

\[
a(k) \equiv \frac{k^2 + k_3^2}{k^2 - k_3^2} Q_{33}(k) - Q_{11}(k) - Q_{22}(k)
\]

\[
= \alpha_{33} q_1(k) \sin^2 \theta
\]

\[
b(k) \equiv \tilde{b}(k) - \frac{1}{4 \pi k^2} \langle \tilde{b}(k) \rangle_k.
\]

\[
= \alpha_{33} q_2(k) \left( \cos^2 \theta - \frac{1}{3} \right)
\]

\[
\langle f(k) \rangle_k = \int_0^\pi d\theta \int_0^{2\pi} d\phi \left[ k^2 \sin \theta f(k) \right]
\]

\[
\cos \theta = \frac{k_3}{k}
\]
Theoretical prediction (1)

- Angular dependence

\[ E_a(k, \theta) \equiv \frac{[a(k)]_{k,\theta}}{[a(k)]_k} = \frac{3}{2} \sin^2 \theta, \]

\[ E_b(k, \theta) \equiv \frac{[b(k)]_{k,\theta}}{[b(k)(3 \cos^2 \theta - 1)]_k} = \frac{5}{4} (3 \cos^2 \theta - 1) \]

\[ [f(k)]_{k,\theta} \equiv \frac{\int_0^{2\pi} d\phi [k \sin \theta f(k)]}{2\pi k \sin \theta}. \]

\[ [f(k)]_k \equiv \langle f(k) \rangle_k / (4\pi k^2) \]
Theoretical prediction (2)

- Scale dependence
  - Assumptions
    - \( q_i(k) \) are functions of only \( \epsilon \) and \( k \) in the inertial subrange

\[
q_i(k) = C_i \epsilon^{2/3} k^{-11/3},
\]

\[
E_a(k) \equiv \langle a(k) \rangle_k := \frac{16}{15} \pi \alpha_{33} C_2 \epsilon^{1/3} k^{-7/3}
\]

\[
E_b(k) \equiv \langle (3 \cos^2 \theta - 1)b(k) \rangle_k = \frac{8}{3} \pi \alpha_{33} C_1 \epsilon^{1/3} k^{-7/3}
\]
Comparison between theory and DNS

- Angular dependence
  - Theory
    \[ E_a(k, \theta) = \frac{3}{2} \sin^2 \theta \]
  - DNS

- for the DNS
  - Every 3°
  - Shells
    - \([k_0, k_1) = [8, 16)\)
    - \([k_0, k_1) = [16, 32)\)
    - \([k_0, k_1) = [32, 64)\)
    - \([k_0, k_1) = [64, 128)\)
    - \([k_0, k_1) = [128, 241)\)

From Ishida & Kaneda (2007)
Comparison between theory and the DNS

- Angular dependence
  - Theory
    \[ E_b(k, \theta) = \frac{5}{4}(3 \cos^2 \theta - 1) \]
  - DNS

From Ishida & Kaneda (2007)
Comparison between theory and DNS

• Scale dependence

Theory

\[
E_a(k) = \frac{8}{3} \pi \alpha_{33} C_1 \epsilon^{1/3} k^{-7/3}
\]

\[
E_b(k) = \frac{16}{15} \pi \alpha_{33} C_2 \epsilon^{1/3} k^{-7/3}
\]

From Ishida & Kaneda (2007)

\[k^{-7/3}/k^{-5/3} = k^{-2/3}\]
From Vorobev et al. (Phys. Fluids 17, 125105 (2005))

\[ c(k) = \frac{E_1(k) + E_2(k)}{2E_3(k)} \]

\[ E_i(k) = \frac{1}{2} \sum_{k-1/2 < |k| < k+1/2} (v_i(k) \cdot v_i^*(k)) , \quad i = 1, 2, 3 \]
Results in our DNS

Anisotropy increases.

approaches to isotropic state. \((c(k) \to 1)\)

From Ishida & Kaneda (2007)
Results in our DNS

Anisotropy increases.

approaches to isotropic state. \((c(k) \rightarrow 1)\)

From Ishida & Kaneda (2007)
• Anisotropy: $c(k)$

  – DNS

  \[
  c(k) = \frac{E_1(k) + E_2(k)}{2E_3(k)}
  \]

  \[
  c_{\text{theory}}(k) = \frac{(8/3)[Q^{(0)}(k) + \alpha_{33}q_3(k)] + \alpha_{33}[(4/15)q_1(k) + (16/15)q_2(k)]}{(8/3)[Q^{(0)}(k) + \alpha_{33}q_3(k)] + 2\alpha_{33}[(16/15)q_1(k) + (4/15)q_2(k)]}
  \]

  \[
  = 1 - \frac{21E_a(k) - 15E_b(k)}{20E(k) + 14E_a(k) - 10E_b(k)}
  \]
Comparison between theory and the DNS

\[ c(k) = \frac{E_1(k) + E_2(k)}{2E_3(k)} \]

\[ k^{-7/3}/k^{-5/3} = k^{-2/3} \]

Comparison with the DNS and the theory

\[ \approx 0.1 \sim 0.2 \]

From Ishida & Kaneda (2007)

T = T_M = 16  magnetic field OFF

T = T_F = 32  magnetic field ON
For turbulence,
We don’t know the pdf nor Hamiltonian in contrast to thermal equilibrium state

But we may assume
1) the existence of equilibrium state at small scale
   (Universal) Local equilibrium state \( a \ la \ K41 \),
   disturbance can be treated as perturbation to the inherent equilibrium state determined by the NS-dynamics

and see
2) the response \( \leftarrow \) small disturbance
disturbances tested:
   Mean Shear, Stratification, MHD
   ( \( \rightarrow \) others, e.g., scalar field under mean scalar gradient).

But
3) The anisotropy may remain large even at small scales,
   if Re is not high enough, so that the inertial subrange is not wide.
Summary - 2

Mean shear
Stratification
MHD

Disturbance; X

Equilibrium state

Response: J

\[ J_{ij} = \Delta Q_{ij} = C_{ijkm} X_{km} \]
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The end