# Two dimensional turbulence

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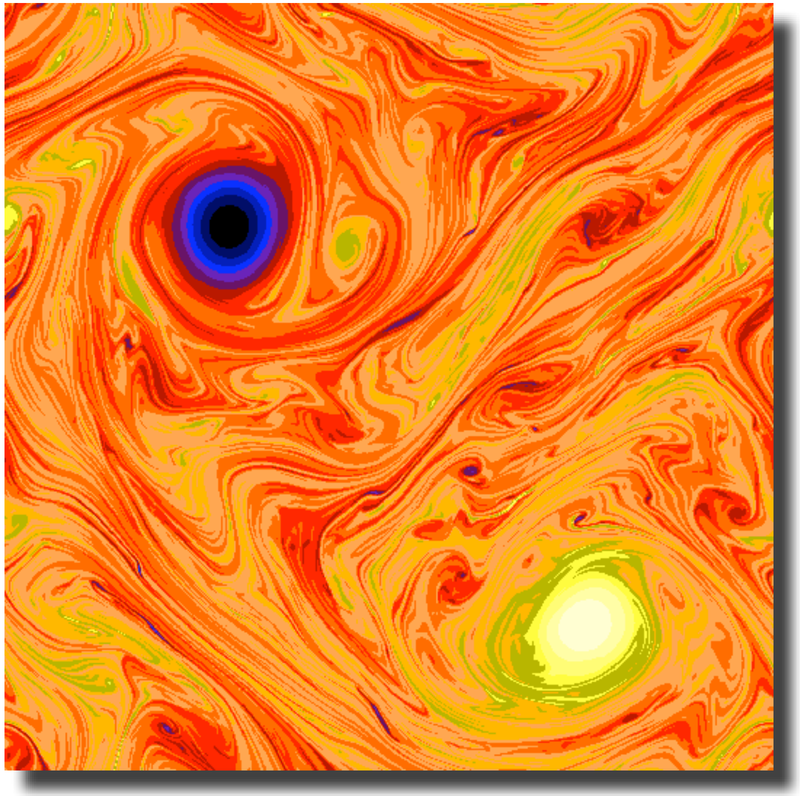
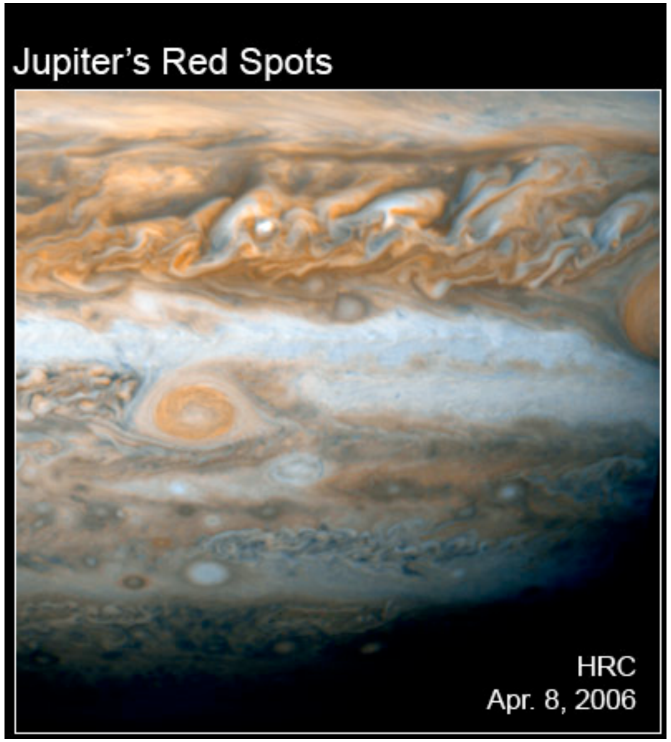
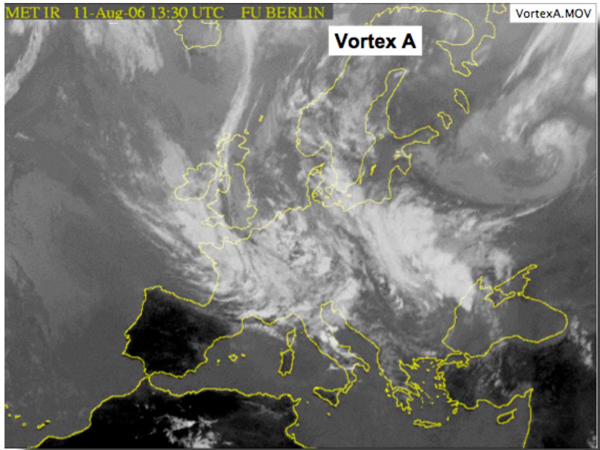


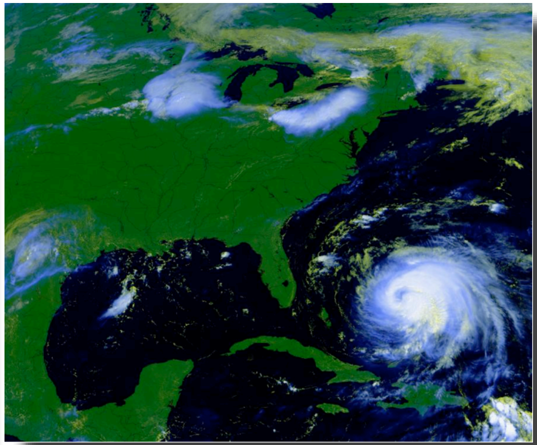
Cargese, August 21, 2007

# Why 2d turbulence?

*Geophysical flows* : mid latitudes large scale flow, stratified atmosphere, shallow layers, tropical hurricanes trajectories, 2d flow frontogenesis, ...



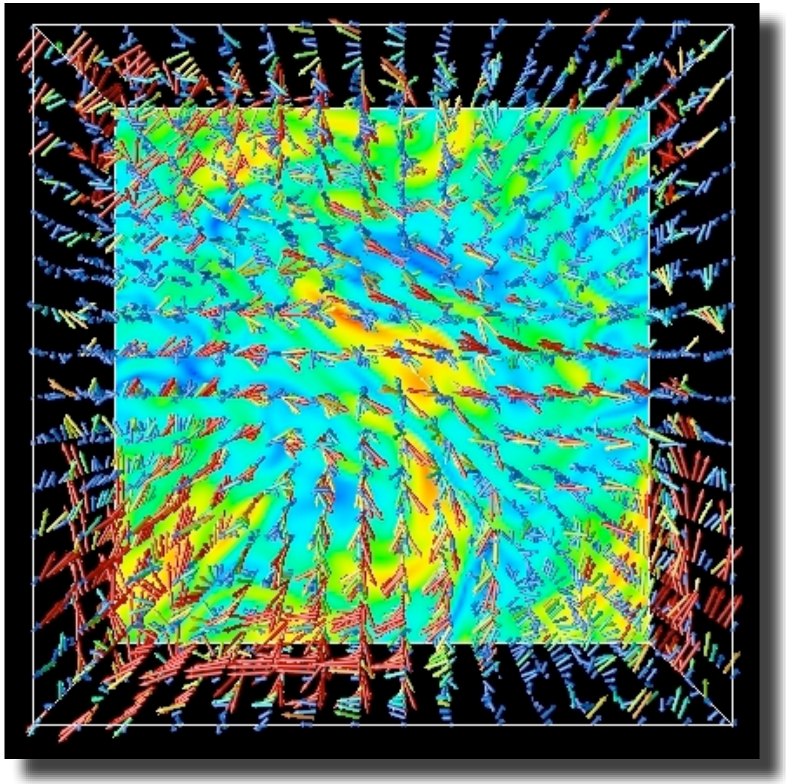



layers of fluid where vertical motion is suppressed by rotation, stratification, confinement

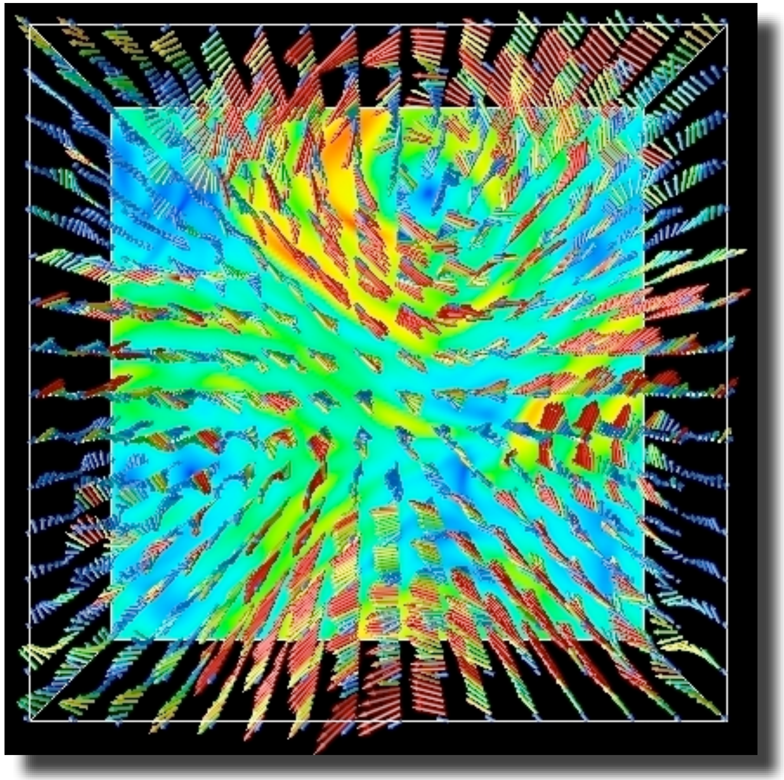
# Other than geophysics..

MHD with external mean axial field $B_0$

weak $B_0$



strong $B_0$

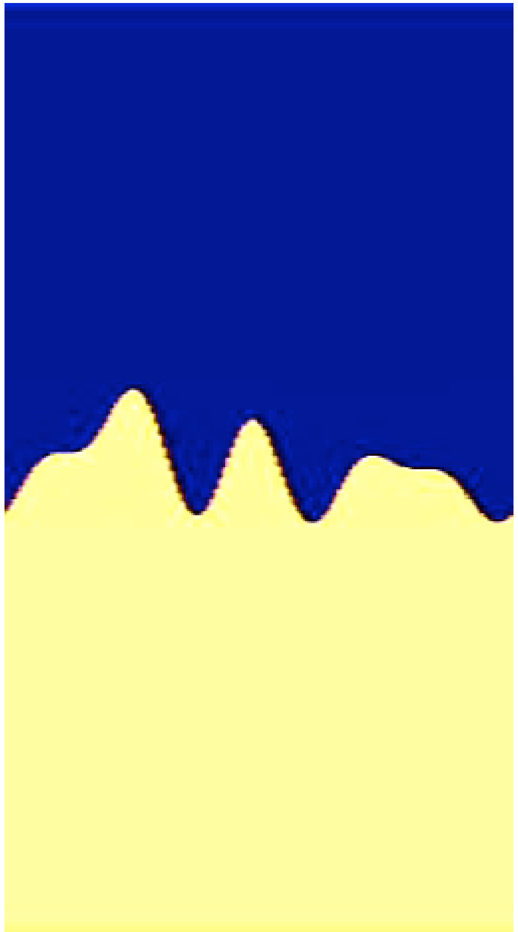


# Practical for studies of *complex or multi-phase turbulent systems*

- Heavy particles advection by 2d flows
- Passive/active scalars advection

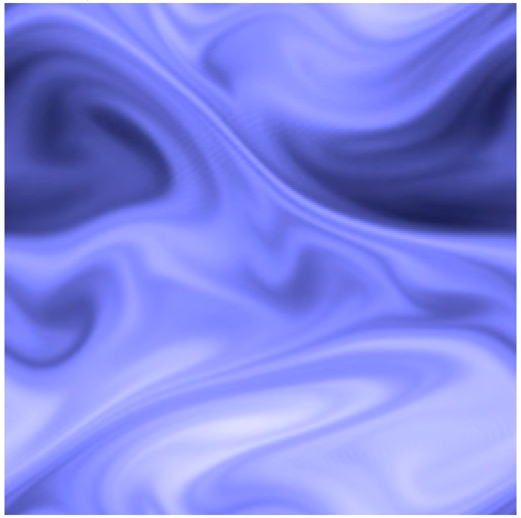
Thanks to L. Vozella, PRL 96, 134504 (2006)

- Rayleigh-Taylor Instability

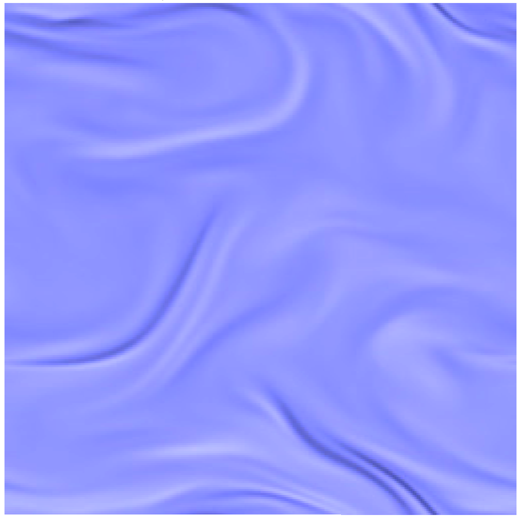


- Drag reduction: Polymers in turbulent flows

no polymer



with polymer



Thanks to S. Musacchio PRL 91, 034501 (2003).

# Ideal set-up for many theories

A very incomplete list *(beyond Kraichnan and Batchelor):*

- Statistical mechanics for conservative systems with many degrees of freedom: Hamiltonian formulation
  (Onsager 1949, Robert & Sommeria 1991, Miller 1990, …)
- 2D Conformal theory
  (Polyakov 1993, Benard et al. 2006)
- Perturbative theories (e.g. Yakhot 2006)
- Closures theories: Eddy Damped Quasi-Normal Markovian approximation, Test Field Model (Orszag, Kraichnan..)
- Quantum field tools for critical phenomena: Renormalization Group

………..

# Literature is very rich..

Andre, Aref, Basdevant, Batchelor, Benzi, Boffetta, Borue, Bouchet, Brachet, Celani, Cenedese, Couder, Dritschell, Ecke, Eyink, Falkovich, Farge, Flor, Frisch, Goldburg, Gollup, Herring, Larcheveque, Legras, Lesieur, Lions, Leith, Lilly, Kraichnan, Marchioro, Meneguzzi, McWilliams, Miller, Montgomery, Nelkin, Onsager, Orszag, Ott, Paret, Politano, Polyakov, Pomeau, Pouquet, Pulvirenti, Rutgers, Sadourny, Santengelo, Saffman, Shneider, Siggia, Smith, Sommeria,Sulem, Tabeling, Tennekes,Van Heijst, Vergassola, Vulpiani, Weiss, Yakhot …………………………………….

Some papers or reference books on the subject:

- Kraichnan, Phys Fluids **10** (1967)
- Kraichnan & Montgomery, Rep. Prog. Phys. **43** (1980)
- Lesieur, "Turbulence in Fluids" (1990)
- Miller et al, Phys Rev A **45** (1992)
- Frisch, "Turbulence" (1995)
- Tabeling, Phys. Rep. **362** (2002)
- Kellay & Goldburg, Rep. Prog. Phys. **65** (2002)

# Outline of the lessons

- Part 1 : Basic notions
- Part 2 : Inverse cascade
- Part 3 : Direct cascade
- Part 4 : A geophysical application
- Few remarks on open (& uncovered) issues and new approaches to the field

# Part 1: Basic Notions

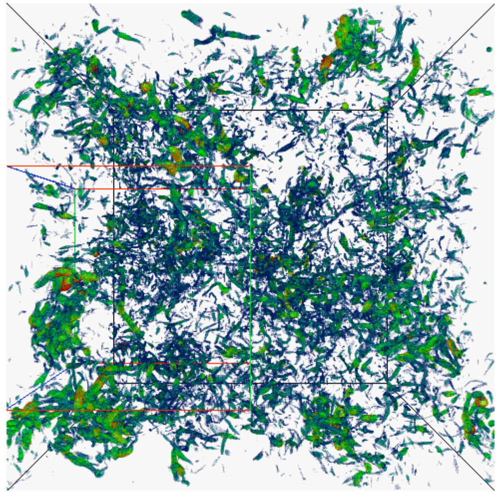
- Equations
- Conserved quantities
- Double cascade scenario

*A half-bubble of about 7 cm diameter on the top of a plastic glass. The system is illuminated by light diffusing from the bottom of the glass. Turbulence is induced from light heating and from air motion around the bubble.*

Thanks to G. Boffetta

# 3d incompressible turbulence

Consider an incompressible flow $\nabla \cdot \mathbf{u} = 0$



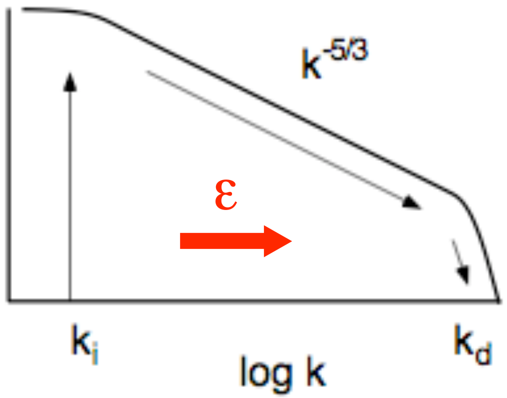
NS eqs. for vorticity $\omega = \nabla \times \mathbf{u}$

$$\partial_t \omega + (\mathbf{u} \cdot \nabla)\omega = (\omega \cdot \nabla)\mathbf{u} + \nu \nabla^2 \omega$$

If $\nu$ =0 energy is conserved $E = \frac{1}{2}\int \mathbf{u}^2 d^2x = \int E(k)\,dk$

If $\nu$ --> 0, Re=UL/$\nu$ --> $\infty$, energy dissipation $\varepsilon = \nu \langle \omega^2 \rangle$ stays finite

*dissipative anomaly*

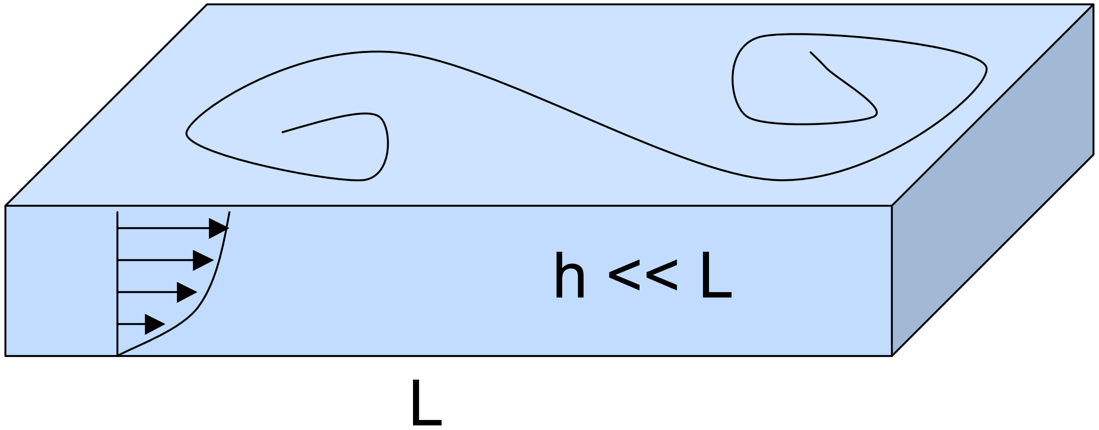



Spectrum follows Kolmogorov behaviour,
but moments of velocity increment
statistics of order p>3 are highly intermittent

$$\langle [\delta_{\|} u(r)]^p \rangle \neq c_p\, r^{p/3}$$

# Ekman-Navier-Stokes turbulence

A shallow layer of incompressible fluid




- vertical motion negligible
  $u_z = O(h/L)\, u_{x,y} \approx 0$
- Poiseuille type velocity profile
  $u_{x,y}(z) \approx z^2$

From 3d Navier-Stokes

$$\partial_t u + (u \cdot \nabla) u = -\nabla p + \nu \nabla^2 u + F$$

$$(\nabla_x^2 + \nabla_y^2 + \nabla_z^2) u \to (\nabla_x^2 + \nabla_y^2) u - \alpha\, u$$

$$\alpha \sim O(\nu / h^2)$$

To 2d Navier-Stokes + friction

$$\partial_t u + (u \cdot \nabla) u = -\nabla p + \nu \nabla^2 u - \alpha u + f$$

$$\alpha \left\{ \begin{array}{l} \text{Ekman friction (rotating)} \\ \text{Rayleigh friction (stratified)} \\ \text{air friction (soap film)} \\ \text{Hartman friction (MHD)} \end{array} \right.$$

# 2d incompressible Navier-Stokes

Consider homogeneous & isotropic 2d flow:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{f} \\ \nabla\cdot\mathbf{u} = 0 \end{cases}$$

$$\mathbf{u} \equiv (\partial_y\psi, -\partial_x\psi)$$

For the scalar vorticity $\omega = \nabla\times\mathbf{u} = -\nabla^2\psi$

$$\boxed{\partial_t\omega + (\mathbf{u}\cdot\nabla)\,\omega = \nu\nabla^2\omega + \nabla\times\mathbf{f}}$$

- At difference with 3d turbulence, no vortex stretching term:

$$(\omega\cdot\nabla)\mathbf{u}$$

- In 2d inviscid unforced flows, a fluid particle conserves its vorticity :

$$\frac{D\omega}{dt} = 0$$

# Conserved quantities & energy/enstrophy balance

- In 2d there are two main inviscid invariants:

Energy $$E = \frac{1}{2}\int \mathbf{u}^2 d^2x = \int E(k)\,dk$$

Enstrophy $$Z = \frac{1}{2}\int w^2 d^2x = \int k^2 E(k)\,dk$$

where
E(k) = energy spectrum

- In the viscous case $\nu>0$ , balance equations are:

$$\frac{dE}{dt} = -\nu Z \qquad \frac{dZ}{dt} = -\nu\langle(\nabla\omega)^2\rangle$$

So, in 2d for Re=UL/$\nu$ $\longrightarrow \infty$ (the viscosity $\nu$ goes to 0)

$$\boxed{\lim_{\nu\to 0}\frac{dE}{dt} = 0} \qquad \boxed{\lim_{\nu\to 0}\frac{dZ}{dt} < 0}$$

No dissipation anomaly for the energy. Energy can not be dissipated at small scale

# Exact results *in the steady case*

In 2d also we have exact relations for the fluxes of energy and enstrophy. These are obtained with a reasoning similar to that leading to the -4/5 law in 3d turbulence:

$$\langle[\delta_{\|}\mathbf{u}(r)]^3\rangle = -\frac{4}{5}\epsilon\, r$$

If the forcing acts at a wavenumber $k_F$ :

- For $k \ll k_F$, energy flux at a constant rate $\epsilon = \nu < \omega^2 >$

$$\langle[\mathbf{u}(\mathbf{x}+\mathbf{r}) + \mathbf{u}(\mathbf{x}) \cdot \hat{r}]^3\rangle = \langle[\delta_{\|}\mathbf{u}(r)]^3\rangle = \frac{3}{2}\epsilon\, r$$

- For $k \gg k_F$, enstrophy flux at a constant rate $\zeta = \nu < (\nabla\omega)^2 >$

$$\langle[\delta_{\|}\mathbf{u}(r)][\delta w(r)]^2\rangle = -\frac{4}{3}\zeta\, r$$

*equivalent of Corssin -Yaglom relation for passive scalar*

# Spectral relations

Energy and enstrophy balance eqs. in Fourier space:

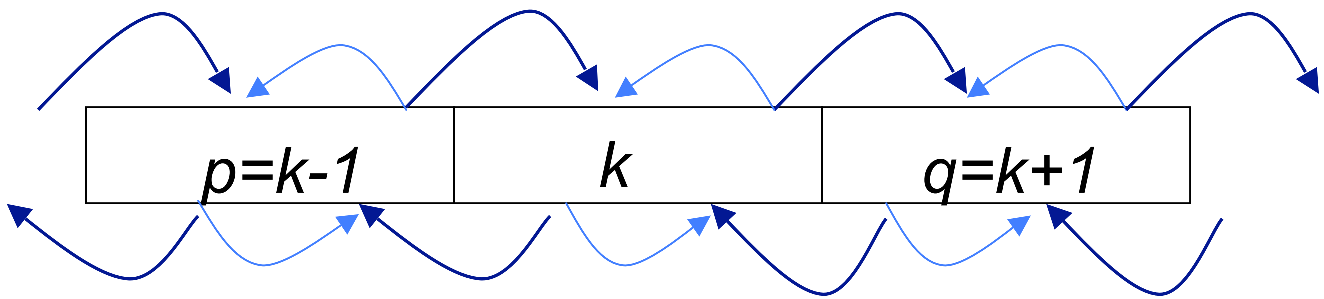
$$(\partial_t + 2\nu k^2)E(k) = T(k) \qquad (\partial_t + 2\nu k^4)E(k) = k^2 T(k)$$

$$T(k) = \int\int dpdq\,T(k,p,q) \sim \int\int_{\mathbf{k}+\mathbf{p}+\mathbf{q}=\mathbf{0}} (..)\, k(k_m\delta_{ij}+k_j\delta_{im})\langle u_i^*(\mathbf{k})u_j(\mathbf{p})u_m(\mathbf{q})\rangle$$

*triad interactions*

Detailed conservation implies:

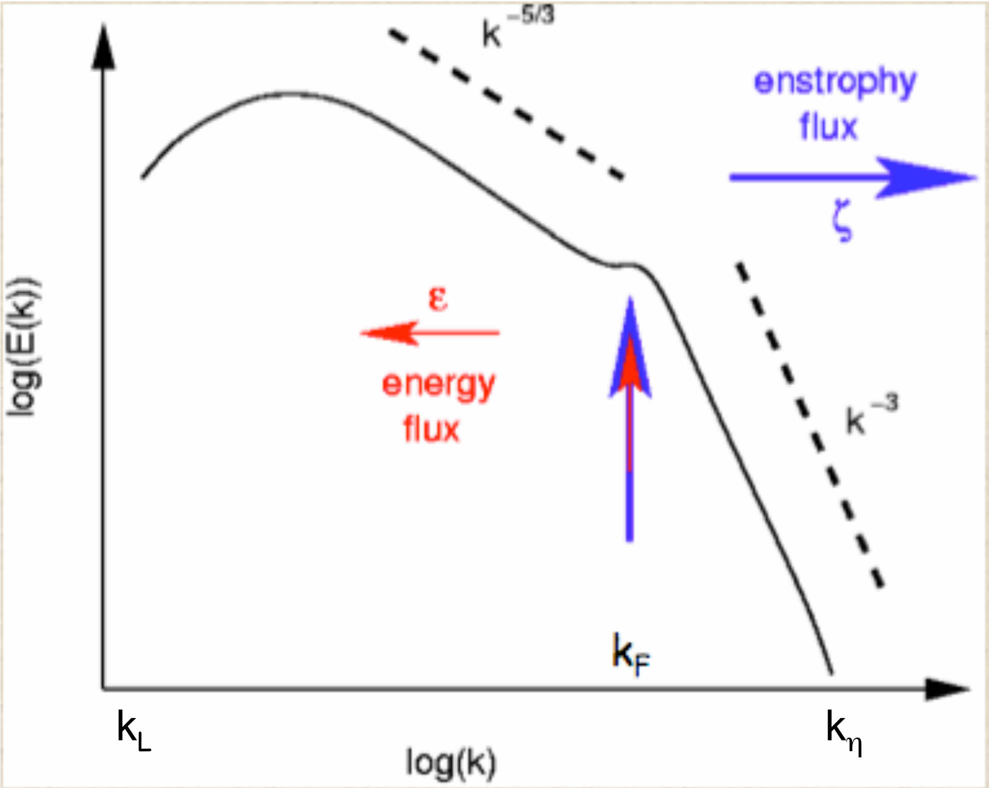
$$T(k,p,q) + T(p,q,k) + T(q,k,p) \equiv 0$$
$$k^2T(k,p,q) + p^2T(p,q,k) + q^2T(q,k,p) \equiv 0$$




+ scaling (constant fluxes) + absolute equilibrium hypothesis.....

# Double cascade scenario

In 1967, **Kraichnan** argued that in the limit of high Reynolds numbers:




$k_F$ is the wavenumber where the forcing F injects energy E and enstrophy Z

Energy goes from the forced scale to larger scales at a constant rate $\varepsilon$ :
$k_L \ll k \ll k_F$ energy inertial range

$$E(k) \sim \epsilon^{2/3} k^{-5/3}$$

Enstrophy goes from the forced scale to smaller scales at a constant rate $\zeta$ :
$k_F \ll k \ll k_\eta$ enstrophy inert. range

$$E(k) \sim \zeta^{2/3} k^{-3}$$

# Dimensional scalings in 2d

We look for (quasi) steady states for the flux of energy and enstrophy.
So two different inertial ranges with self-similar behaviours..

For $\eta << r << r_F$, 2d enstrophy cascade

$$\langle [\delta_{\|} \mathbf{u}(r)][\delta w(r)]^2 \rangle = -\frac{4}{3}\zeta\, r$$

$$\downarrow$$

$$\zeta \sim \left(\frac{\delta\omega^2(r)}{\tau_r}\right) \sim \frac{1}{r}\langle\, \delta u(r) \left(\frac{\delta u(r)}{r}\right)^2 \rangle$$

$$\downarrow$$

$$\boxed{\delta u(r) \sim \zeta^{1/3}\, r; \quad \delta\omega(r) \sim \zeta^{1/3}}$$

$$\boxed{E(k) \sim \zeta^{2/3} k^{-3} \ln^{-1/3}(k/k_F)}$$

Kraichnan, JFM **47** (1971)

For $r_F << r << r_L$, 2d energy cascade

$$\langle [\delta_{\|} \mathbf{u}(r)]^3 \rangle = \frac{3}{2}\epsilon\, r$$

$$\downarrow$$

$$\epsilon \sim \frac{\delta u^2(r)}{\tau_r} \sim \frac{\delta u^3(r)}{r}$$

$$\downarrow$$

$$\boxed{\delta_{\|} u(r)] \sim (\epsilon r)^{1/3}}$$

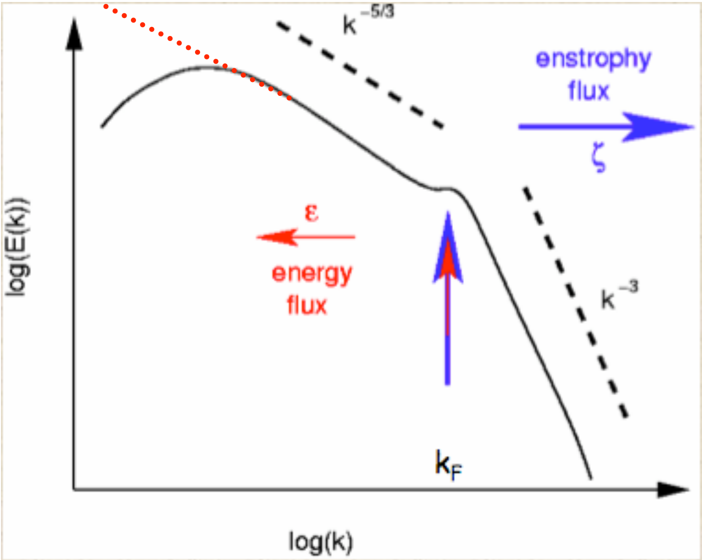
$$\boxed{E(k) \sim \epsilon^{2/3} k^{-5/3}}$$

# Before Exp & DNS results, two remarks

- Stationarity

Enstrophy is dissipated at small scales $\eta \sim \left(\frac{\nu^3}{\zeta}\right)^{1/6}$

What happens to energy at large scales?
Kolmorogorov like inertial scaling is stationary only if a sink of energy is added at large scales, for example a term "$-\alpha u$" or "$\Delta^{-p} u$" in the eq.




- Scales interactions

In the inverse cascade, hierarchy of eddies

$$\tau_r \simeq r^{2/3}$$

In the direct cascade, only one typical time $\tau \simeq 1/ < \omega_{rms} >$
Large & small eddies can interact, no time decorrelation mechanism

# Measurements in atmospherical flows

Mesoscale wind variability (below 5km) (radar and balloon measurements): Evidences of E(k)= k ^(-5/3) spectrum

*Gage, J. Atmos.Sciences **36** (1979)*

Global Atmospheric Sampling Program (GASP) aircraft measurements:
Good evidences of E(k)= k ^(-5/3) spectrum between 3-300 km
But evidences of k^(-3) spectrum at larger scales

*Nastrom, Gage, Jasperson, Nature **310** (1984)*

Horizontal wavelenght

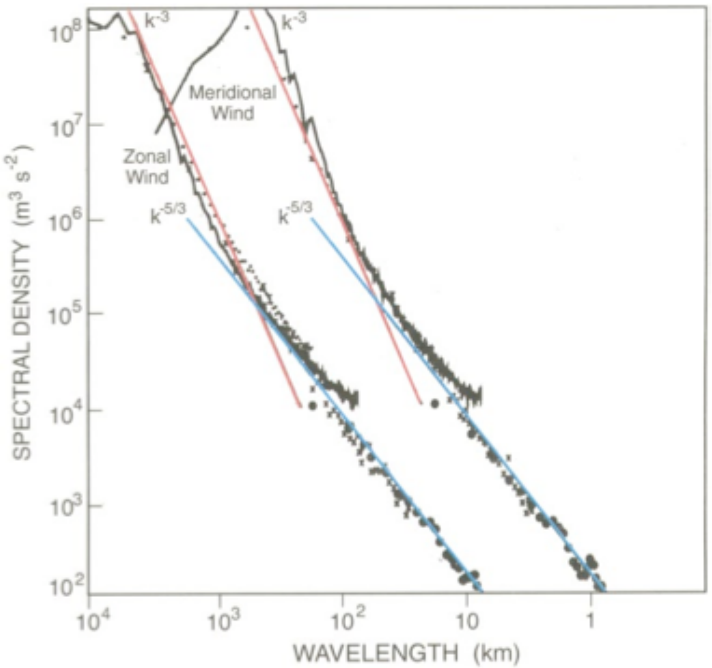



FIG. 1. Zonal wavenumber spectra of the zonal and meridional wind components in the upper troposphere. The data points are reproduced from Nastrom et al. (1984) and show actual observations based on data from commercial aircraft flights, with different symbols representing results obtained using different lengths of flight segments. The straight lines are drawn for reference and have slopes of −5/3 and −3. The solid curve is for the N270L40 SKYHI model along the 45°N latitude circle and at 211 mb, monthly averaged for a single Jul. For clarity the results for the meridional wind have been shifted one decade to the right.

*However things can be complicated by the presence of gravity waves*

# Laboratory experiments: conducting flows

Thin layers of conducting fluid driven by electric current

*(perpendicular magnetic field dumps vertical motion)*

(Bondarenko et al, <u>*Sommeria*</u>, Tabeling, Gollup, Cenedese,..)

Electric & PIV measurements

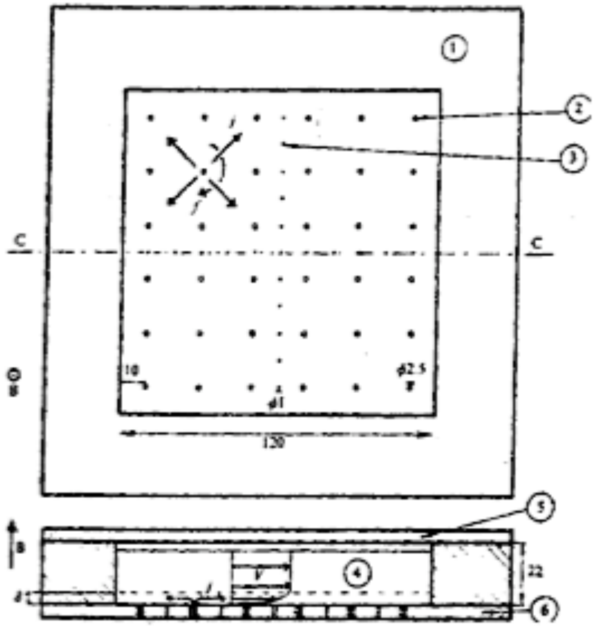


**Figure 5.** The apparatus: the current distribution near one electrode and the velocity profile are schematized. The Hartmann layer depth is denoted by $\delta$. (1) Copper frame. (2) Electrodes for current injection and electric potential measurements. (3) Electrodes for electric potential measurements only. (4) Mercury. (5) Glass cover. (6) Electrically insulating bottom plate in which electrodes are embedded.

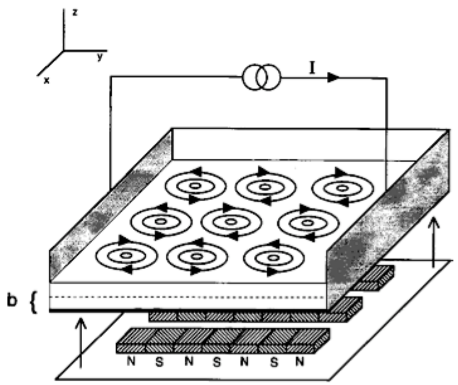
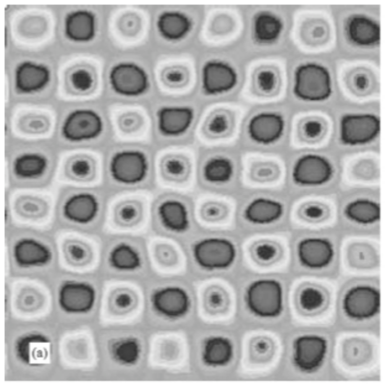
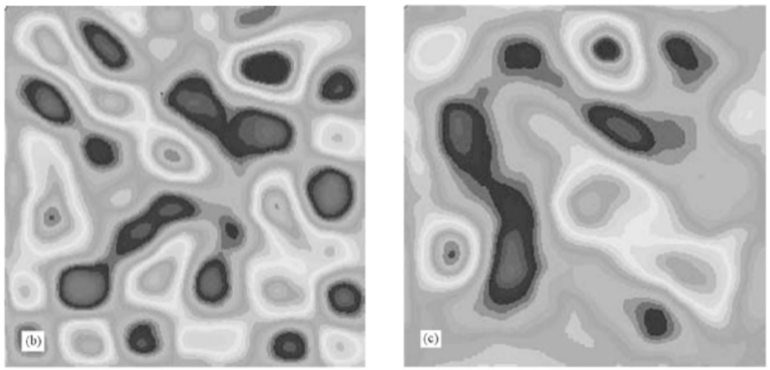



FIG. 1. The experimental set-up.



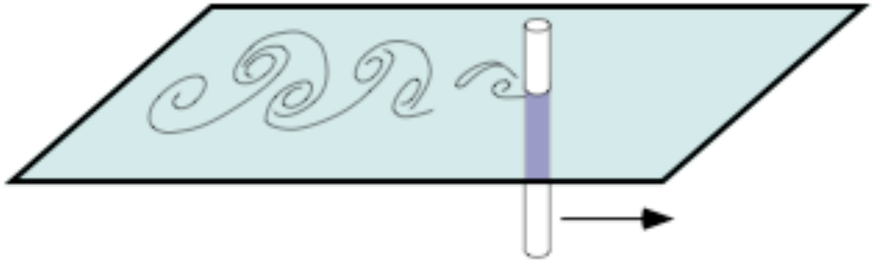




*Decay exp. : time --->*

# Laboratory experiments: soap film

**Soap films**
(<u>*Couder*</u>, Gharib, Derango, Goldburg, Kellay, Bruneau, Rutgers, ….)

A layer of μm, in the plane eddies of size up to cm



Couder set up:
Decay turbulence

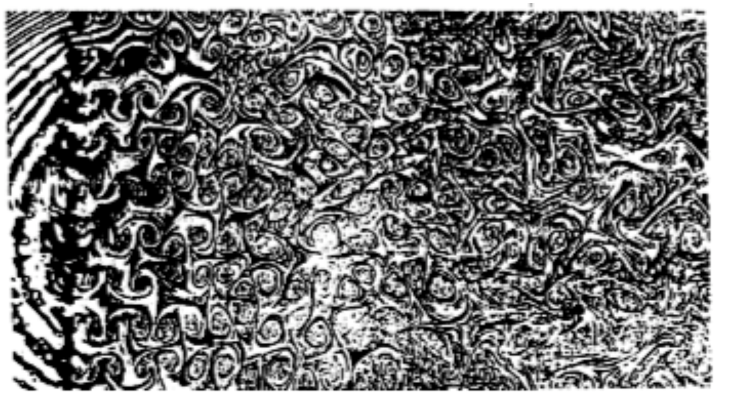
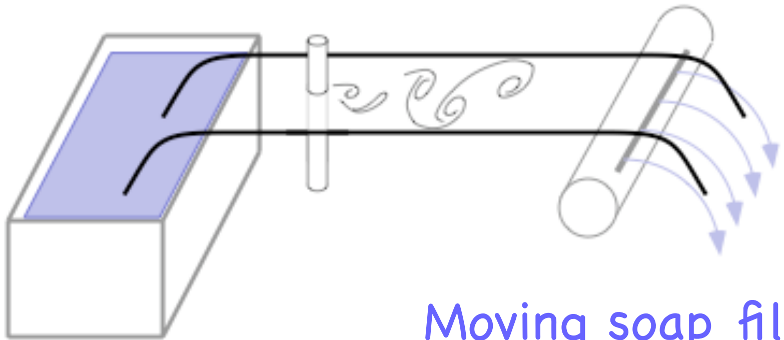


**Figure 2.** Grid generated turbulence obtained by towing an array of cylinders through a still horizontal soap film.



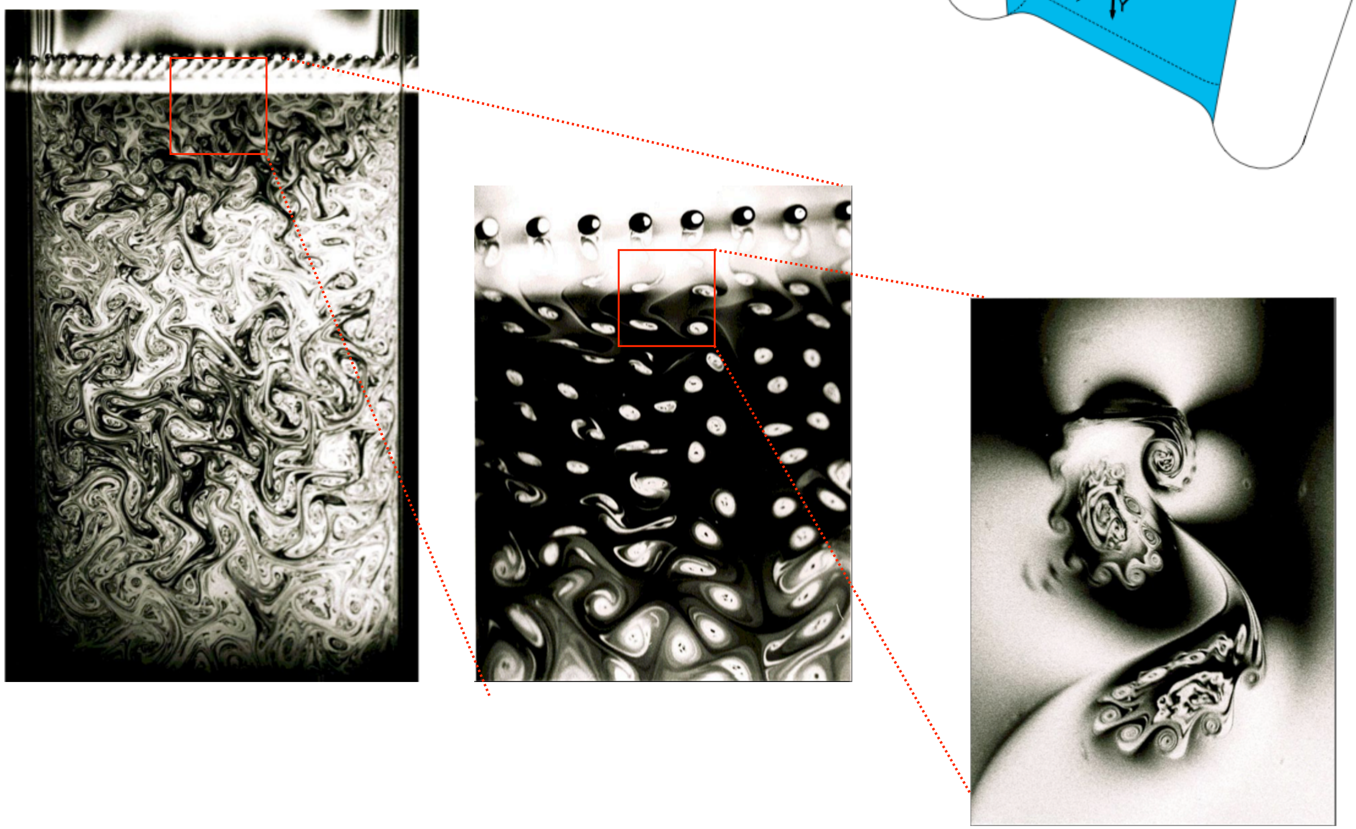
Moving soap film

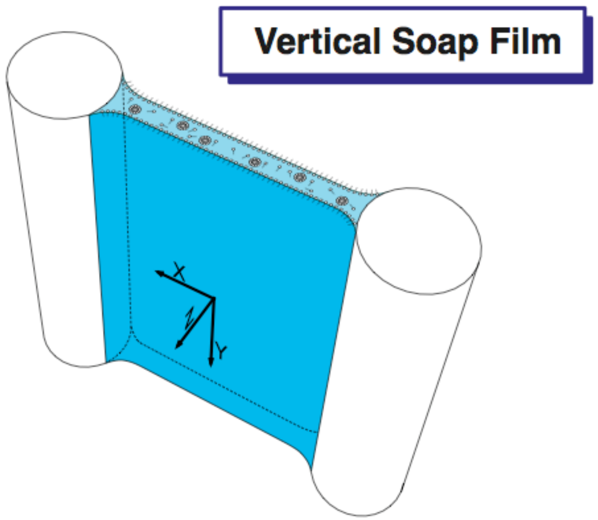
M. Gharib, and P. Derango, Physica **D37**, 406 (1989).

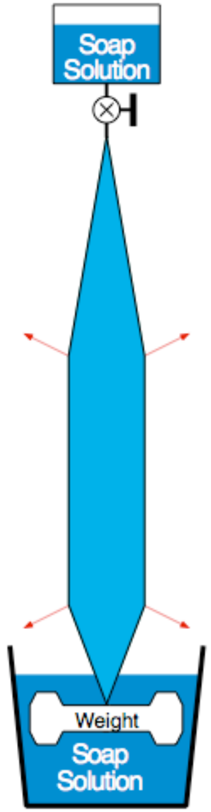
LDV visualization

# Vertical soap film

PIV, LDV, interferometry
measurements



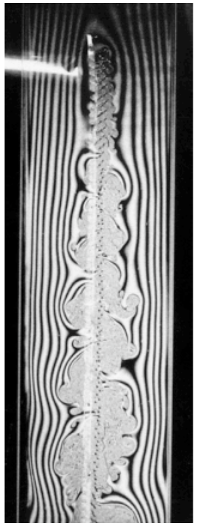
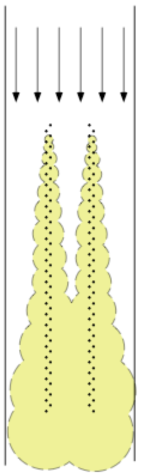



*99% water*
*+ a detergent*

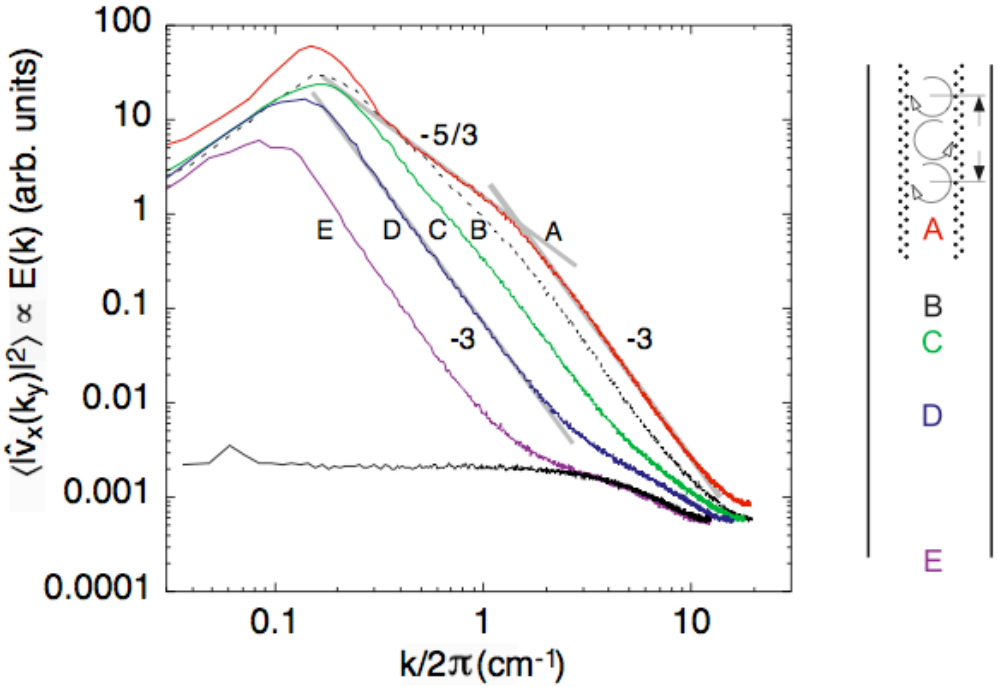
# Experimental steady double cascade

Vertical soap film, with vertical combs (1d grid)


1 central vertical comb
Or two vertical combs as $\Lambda$

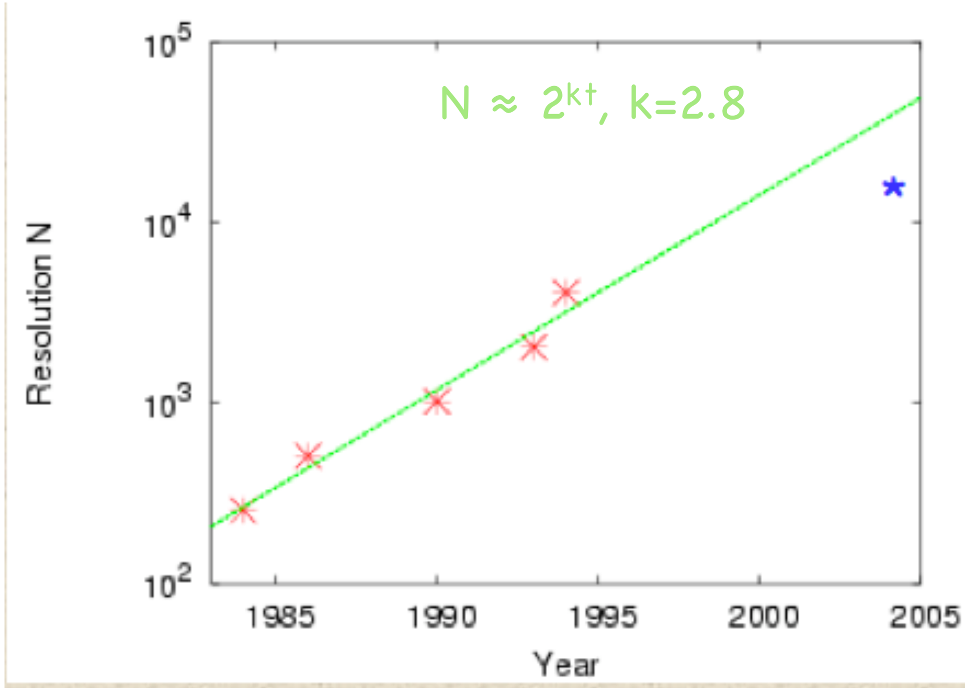
Varying measurement position,
from forced to decay 2d turbulence




M. A. Rutgers, PRL 81, (1998)

# DNS: historically

Reasonable resolutions DNS
started with
Frisch & Sulem at $256^2$
(Phys. Fluids **27** (1984))
to get to Borue at $4096^2$
(PRL **71** (1993))




Thanks to G. Boffetta

Standard numerical approach:

*Pseudo-spectral methods, parallel computing*

$$\partial_t \omega + (\mathbf{u} \cdot \nabla)\, \omega = \nu \nabla^2 \omega + \nabla \times \mathbf{f}$$

*Square box with periodic BC* (unless role of boundaries is of interest)

# High-resolution DNS (I)

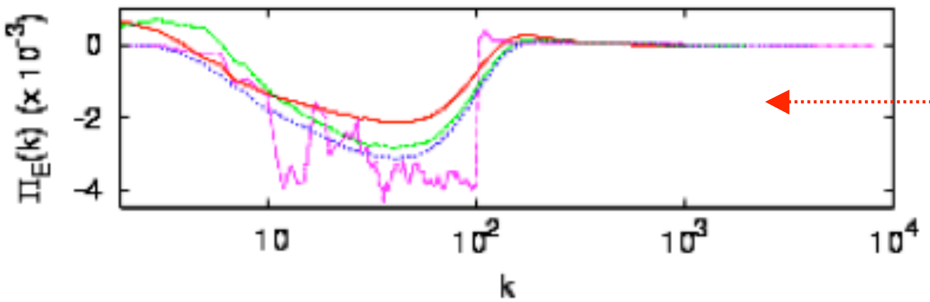
2d NS + linear friction: $$\partial_t \omega + (\mathbf{u} \cdot \nabla)\, \omega = \nu \nabla^2 \omega - \alpha \omega - \Delta f$$

*(Boffetta, nlin.0612035)*

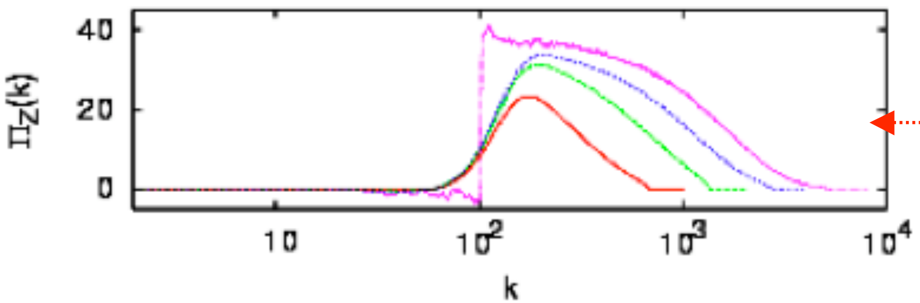
Simulations are designed so to:

- keep constant the ratio $L/l_F$ at increasing resol. N
- keep constant friction scale at increasing $\varepsilon_\alpha = \varepsilon_{in} - \varepsilon_\nu$ : $l_\alpha \sim (\epsilon_\alpha/\alpha^3)^{1/2}$

| $N$ | $\nu$ | $\alpha$ | $\ell_f/\ell_d$ | $\varepsilon_\alpha/\varepsilon_I$ | $\varepsilon_\nu/\varepsilon_I$ | $\eta_\alpha/\eta_I$ | $\eta_\nu/\eta_I$ |
|---|---|---|---|---|---|---|---|
| 2048 | $2 \times 10^{-5}$ | 0.015 | 26.2 | 0.54 | 0.46 | 0.03 | 0.97 |
| 4096 | $5 \times 10^{-6}$ | 0.024 | 52.3 | 0.82 | 0.18 | 0.08 | 0.92 |
| 8192 | $2 \times 10^{-6}$ | 0.025 | 80.5 | 0.92 | 0.08 | 0.10 | 0.90 |
| 16384 | $1 \times 10^{-6}$ | 0.03 | 114.2 | 0.95 | 0.05 | 0.12 | 0.88 |



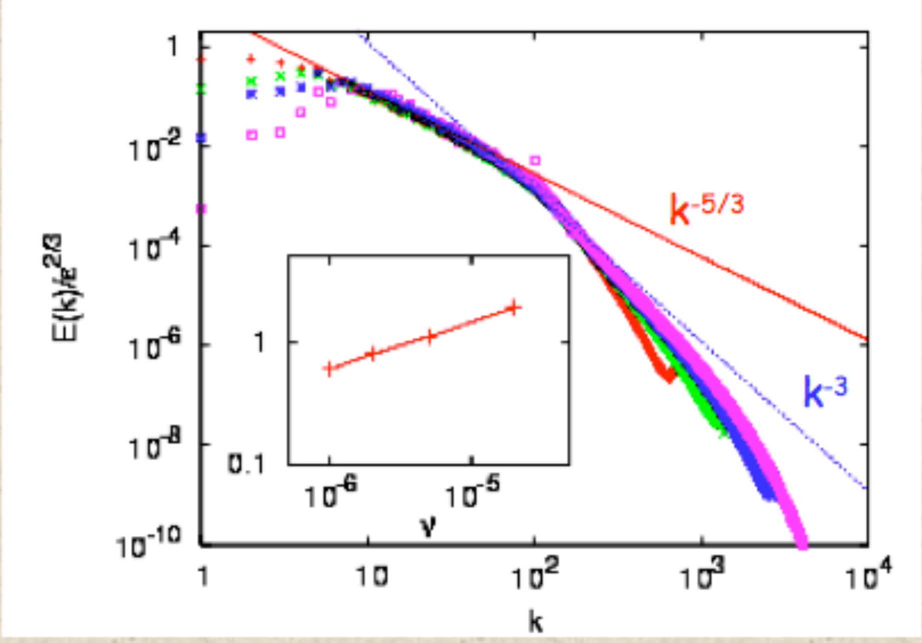

*Energy inertial range indip. of small scale viscosity*




*Enstrophy inertial range increases, as expected, with N*

* Same forcing except DNS at larger N

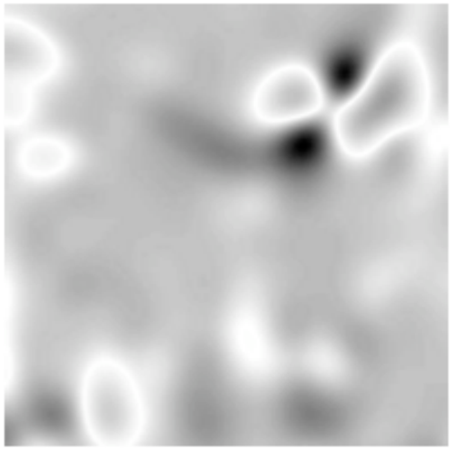
# High-resolution DNS (II)

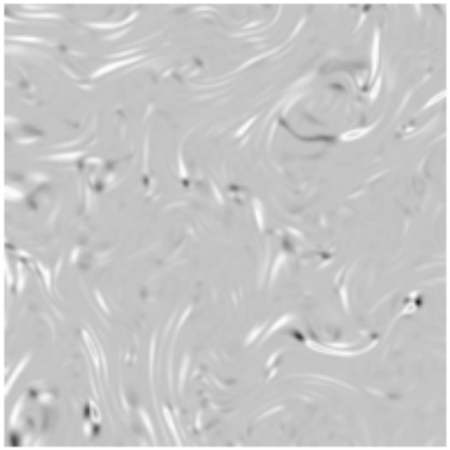


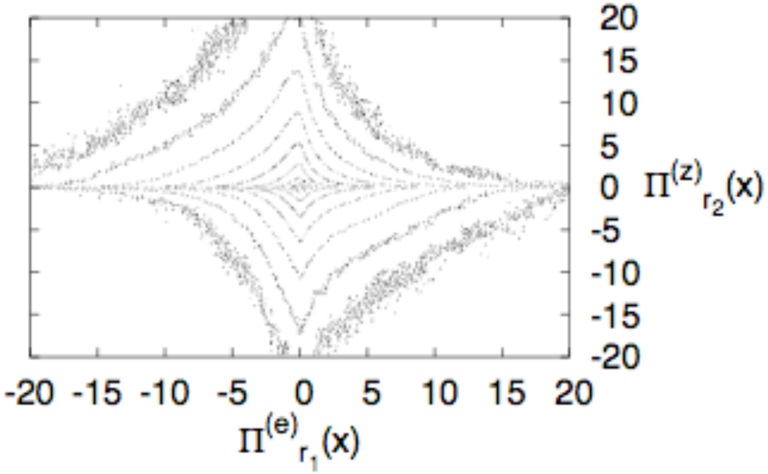

- Corrections to $E(k) \approx k^{-3}$ spectrum decrease with increasing N

  (to overcome finite size effects both $\alpha$ and $\nu$ should possibly go to zero)

- Locally both fluxes are positive and negative; independence of cascades (at the scales where they are maximal, very poorly correlated)



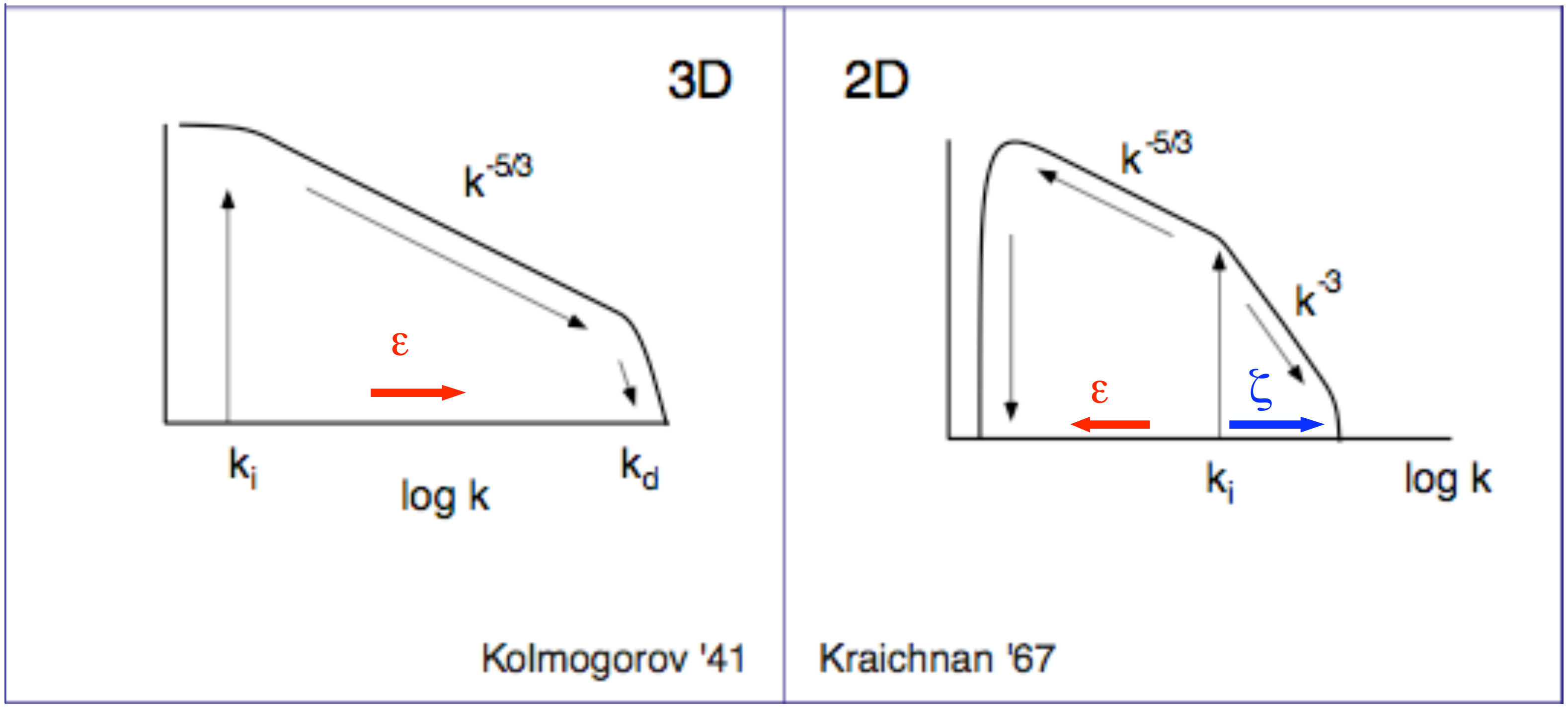





Physical space
energy flux, r1=0.025L

Physical space
enstrophy flux, r2=0.0025L

# Summary


3D
k-5/3
ε
ki
log k
kd
Kolmogorov '41
2D
k-5/3
k-3
ε
ζ
ki
log k
Kraichnan '67

# Part 2: energy inverse cascade

- Experimental & numerical results

- Discussion of main statistical properties :

  *absence of intermittency; almost Gaussian statistics*

Early studies:

. *Frisch & Sulem, DNS 1984*

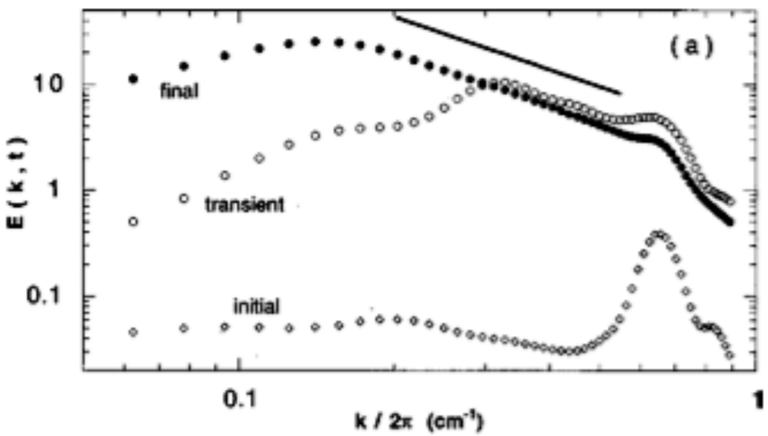
. *Sommeria, electrolyte solution 1986*

# Some observations

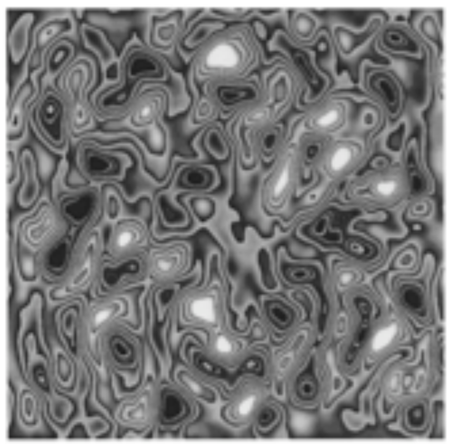
Thin electrolyte cell (L x L) : stationary state is assured by Hartman friction

Energy inertial range : $l_F \ll r \ll l_\alpha < L$

Paret & Tabeling, Phys Fluids 10, 1998




energy spectrum



vorticity

vorticity evolution
to the steady state

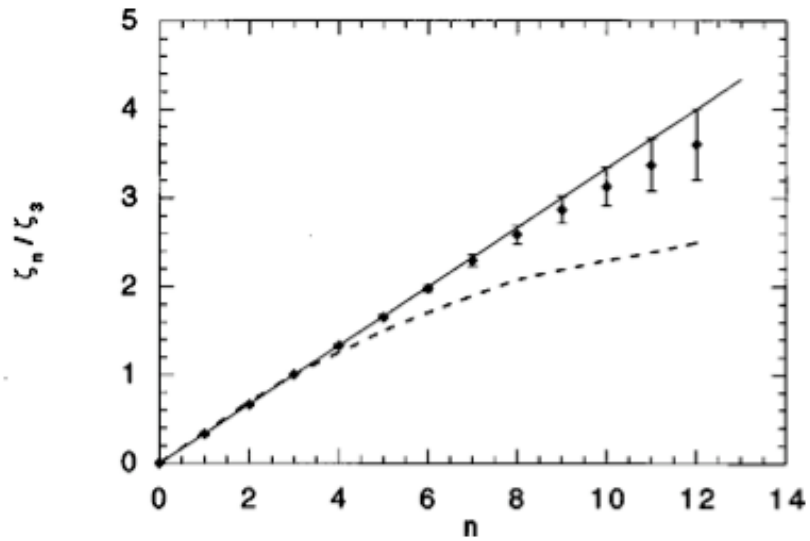
- there are no vortices larger than the injection scale (from vortex size distrib.)
  --> *no vortices merging, very different from decaying 2d turbulence*
- rather, aggregation in clusters of small-size equal sign vortices is the transfer mechanism of energy to larger scales

# What do we expect ?

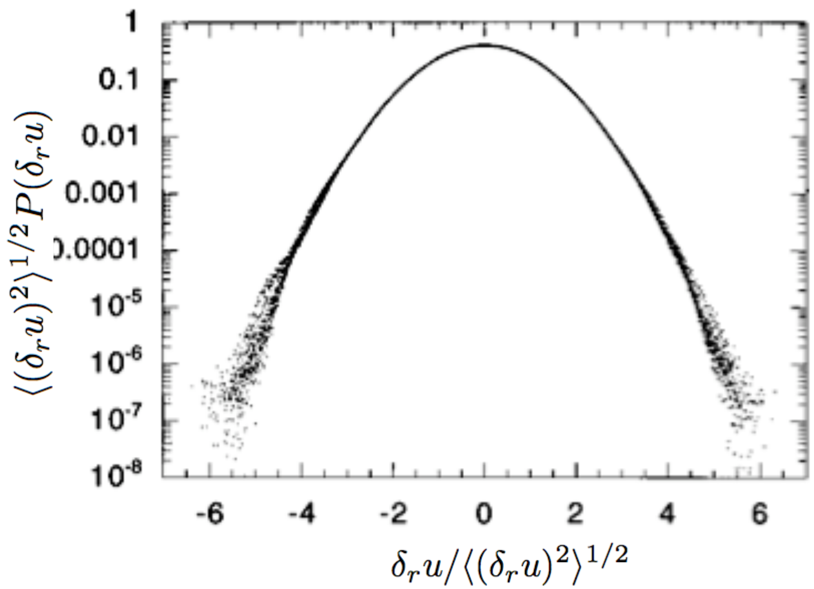
From Kraichnan 1967,
+ DNS (Smith&Yakhot 1993-94) + EXP. (Paret&Tabeling 1998):

- Kolmogorov type spectra $E(k) \approx k^{-5/3}$
- energy dissipation vanishing for viscosity --> 0;
  no dissipative anomaly
- no intermittency corrections in velocity statistics

$$S_n(r) = \langle [\delta_{\parallel} u(r)]^n \rangle = c_n \, \epsilon^{n/3} r^{\zeta_n}$$




Paret & Tabeling, Phys Fluids 10, 1998



# Questions

- Can we conclude that velocity statistics is (almost) Gaussian in the inverse turbulent cascade?

- What about our understanding of non-linear terms (associated to a constant flux) in NS eqs.

  and deviations from Gaussian statistics?

- Is the statistics universal (i.e. forcing independent) ?

  in which sense?

# DNS of inverse cascade

Boffetta, Celani, Vergassola PRE 61, 2000

$$\boxed{\partial_t \omega + (\mathbf{u} \cdot \nabla)\, \omega = \nu \nabla^2 \omega - \alpha \omega - \Delta f}$$

A convenient forcing is the following: *Gaussian & white-in-time*

$$\begin{cases} \langle f(\mathbf{r},t) f(\mathbf{r}',t') \rangle = F_0 l_F^2 \, \exp[-(\mathbf{r}-\mathbf{r}')^2/(2l_F^2)] \, \delta(t-t') \\ \epsilon_{in} = F_0 \end{cases}$$

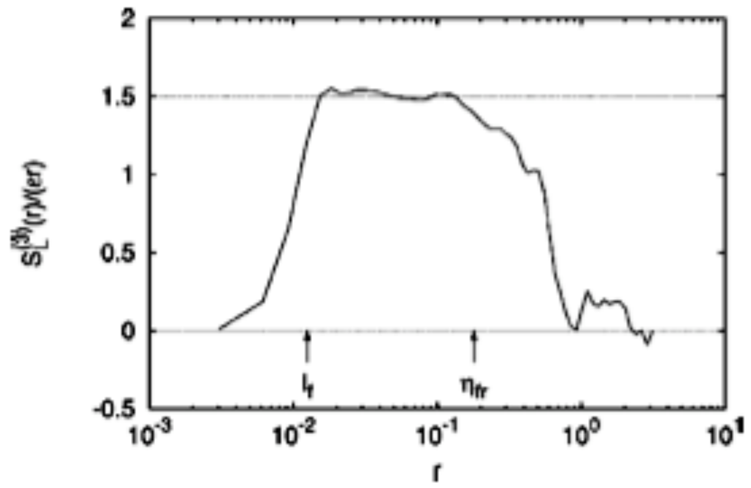
*energy input*

- Resolution is $N^2=2048^2$, *pseudo-spectral methods*
- Statistics is over 80 snapshots (one each $T_L$)

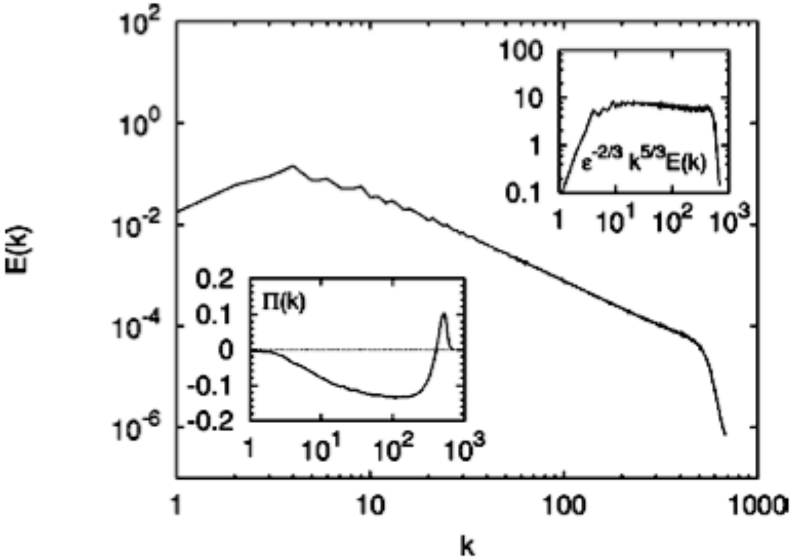
- Friction scale $l_\alpha \ll L$ the box size
- inertial range $l_F \ll r \ll l_\alpha$

- 2 different forcings to probe universality issues

# Results for flux & spectrum




- $\langle[\mathbf{u}(\mathbf{x}+\mathbf{r})+\mathbf{u}(\mathbf{x})\cdot\hat{r}]^3\rangle = \frac{3}{2}\epsilon\, r$




- $E(k) = C\epsilon^{2/3}\,k^{-5/3}$

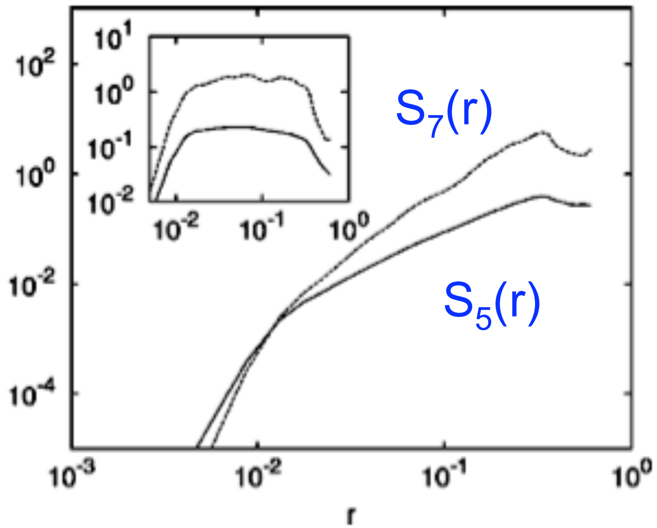
$C = 6. \pm 0.4$ Universal value

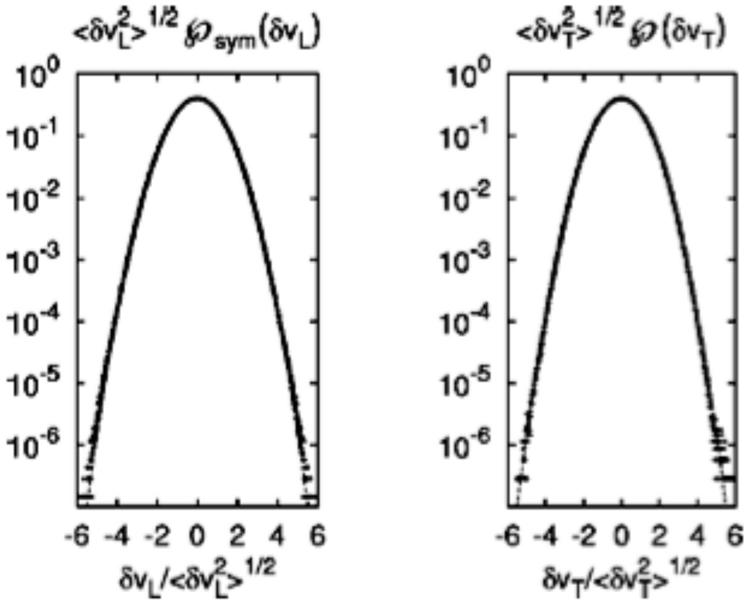
Deviations might appear at low k if different large scale frictions are used

# High order statistics

Even and odd order longit. moments scale dimensionally: no intermittency

$$S_n(r) = \langle [\delta_{\|} u(r)]^n \rangle = c_n (\epsilon r)^{n/3}$$




PDF long increments $P(\delta_{\|}u(r))$ & PDF trasv increments $P(\delta_t u(r))$ are almost Gaussian

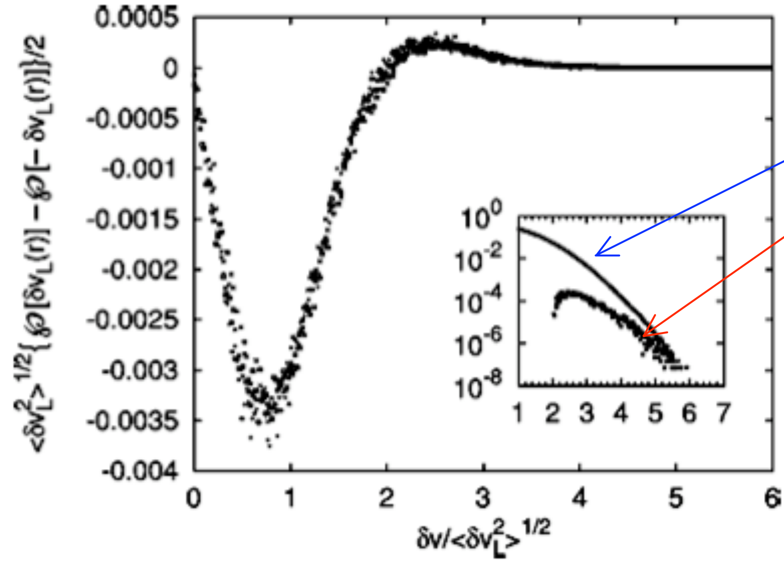
$$\mathcal{S} = \frac{\langle (\delta_{\|} u(r))^3 \rangle}{\langle (\delta_{\|} u(r))^2 \rangle^{3/2}} = 0.03$$

Skewness is small, Hyperskewness can be high

# Departures from Gaussianity

To have a deeper insight, measure the antisymmetric PDF

$$P^{antisym} = P(\delta_{||}u(r)) \ - \ P( - \delta_{||}u(r))$$




Symmetric tail

Antisymm. tail

Asymmetries, altough small, are not negligible for large fluctuations

Odd order statistics and antisymmetric PDF appear universal

Even high order stat. and symmetric PDF appear more forcing dependent

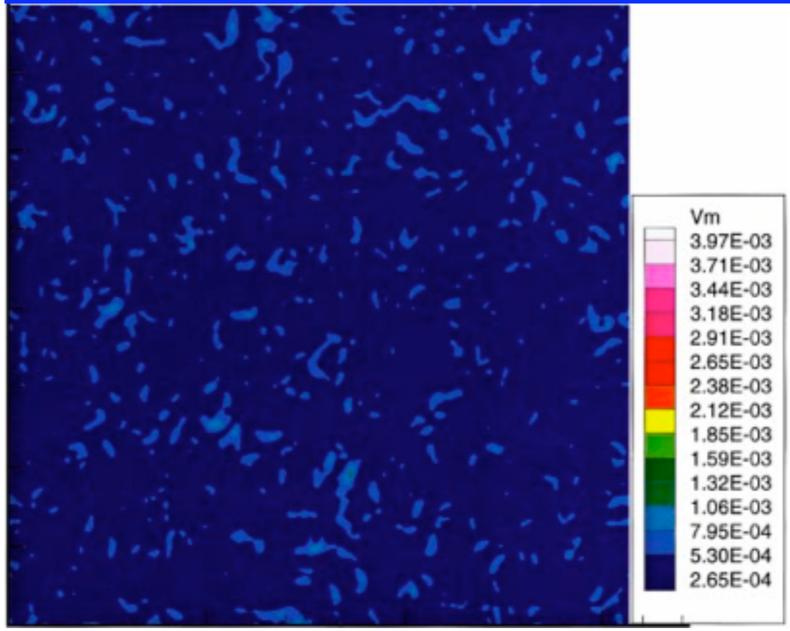
# Summary

A 2d flow with energy injected at $l_F$ and removed at $l_\alpha \ll L$ exhibits an inverse cascade of energy characterized as:

- an energy spectrum as predicted by Kraichnan
- no intermittency corrections to dimensional scaling
- universality holds but for some observables
- small but detectable deviations from Gaussian stat.
- closure theory results compatible with numerics
  (Yakhot PRE 60, 1999)

Paret & Tabling, Phys Fluids 10, 1998
Boffetta, Celani, Vergassola PRE 61, 2000

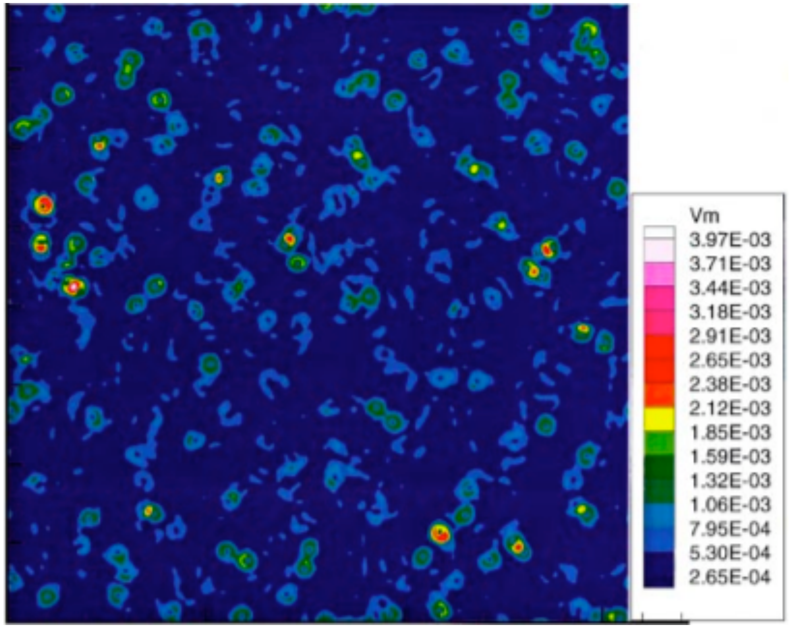
# What happens with small or no large-scale friction ?




- *No formation of structures at scales $r > l_F$, if large-scale drag is properly parametrized*

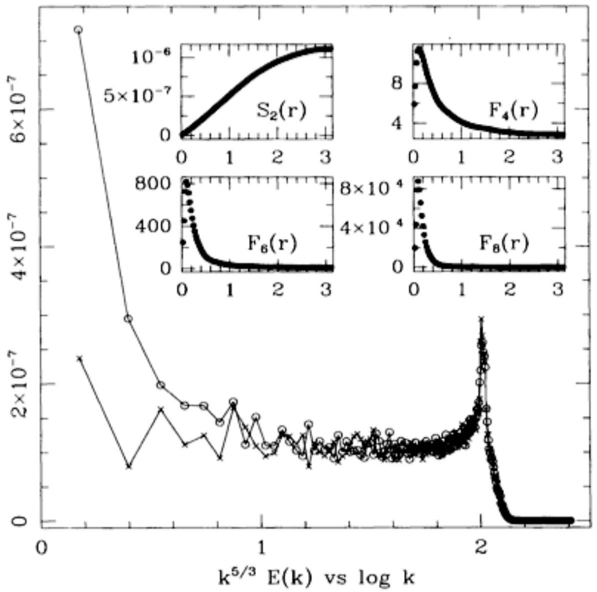
Sukoriansky et al, Phys Fluids 11, 1999

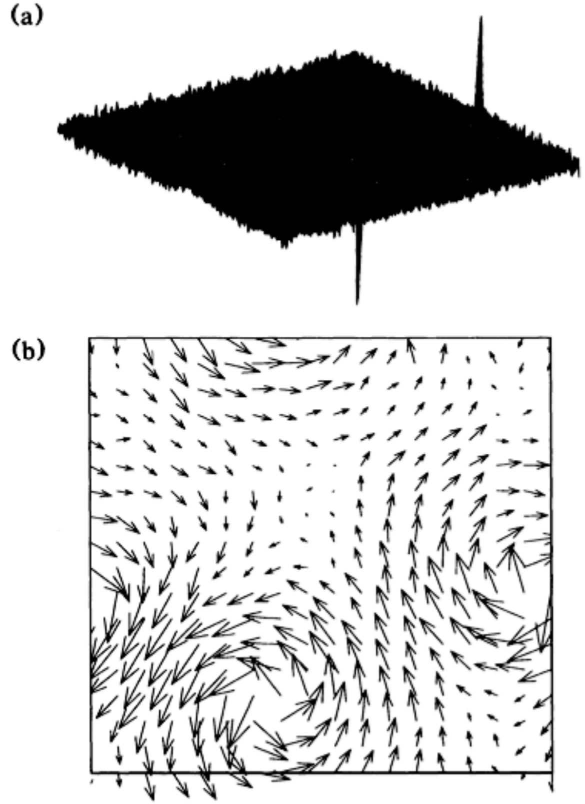
- *Otherwise, we have the formation of structures at scales $r > l_F$*



# Bose-Einstein condensation of energy

If the friction scale is larger than the size of the system $l_\alpha > L$
energy accumulates at the largest possible scale
*(in the smallest possible mode $k_0=1/L$ )*




vorticity
"dipole"

velocity
field

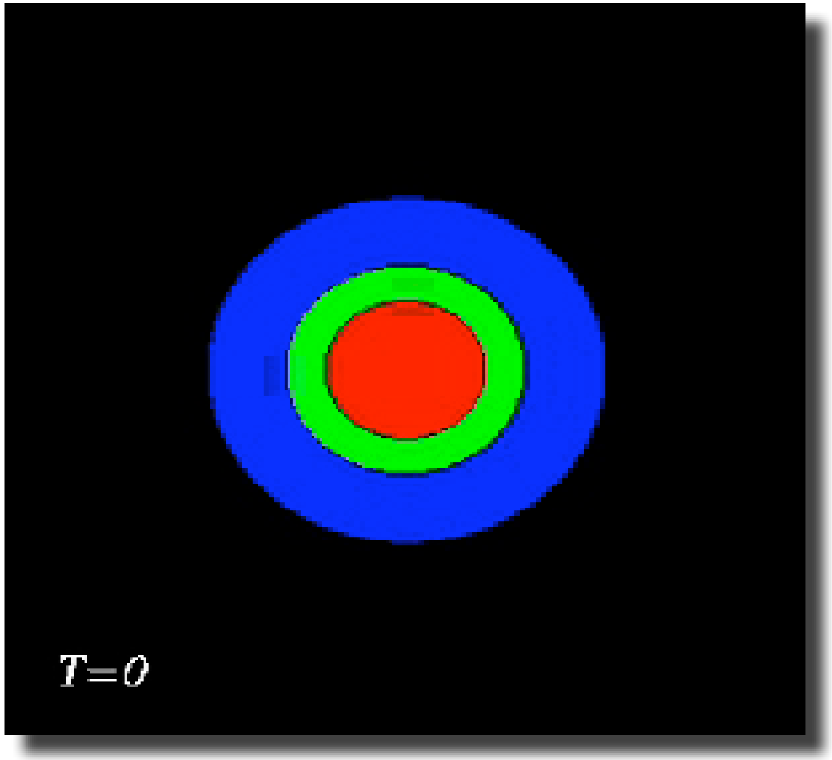
- Strong deviations from $E(k) \approx k^{-5/3}$
- Strong deviations from non intermittent stat.

*(Kraichnan 1967, Smith & Yakhot 1993, see also Borue 1994)*

# Part 3: enstrophy direct cascade

- Experimental & numerical results

- Vorticity & velocity statistics




*Early works:*
*Kraichnan 1967, Batchelor 1969*

# Recall basic features

Energy is exchanged between modes but there is not a cascade: *E-spectrum $E(k) \approx k^{-3}$*

Enstrophy $Z \approx \langle \omega^2 \rangle$ cascades at a constant rate $\zeta$ from large to small scales *(elongation of vorticity patches):*

*Z-spectrum $Z(k) \approx k^{-1}$*

Eddies are not organized hierarchically in time: at any scale r, we have the same eddy turn-over-time

$$\tau \simeq 1/ < \omega_{rms} >$$

# Early observations *(mainly numerics)*

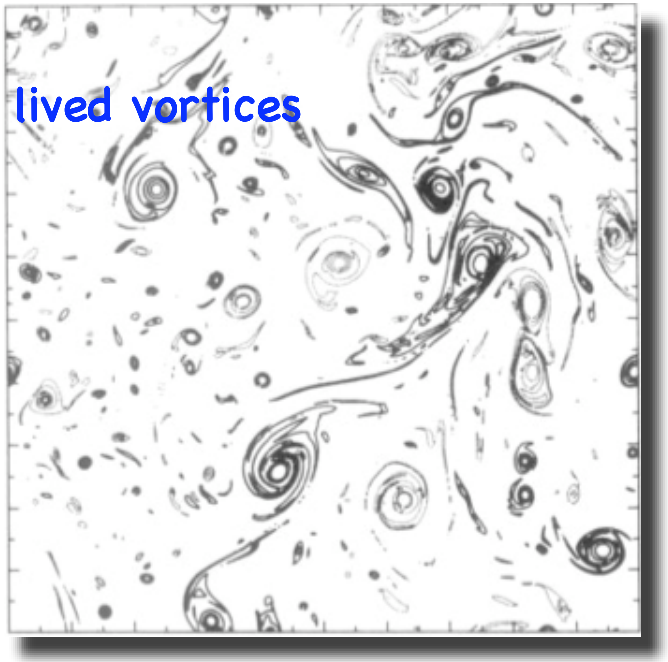
First observations report deviations from the spectrum predicted by theory: $E(k) \approx k^{-3}$

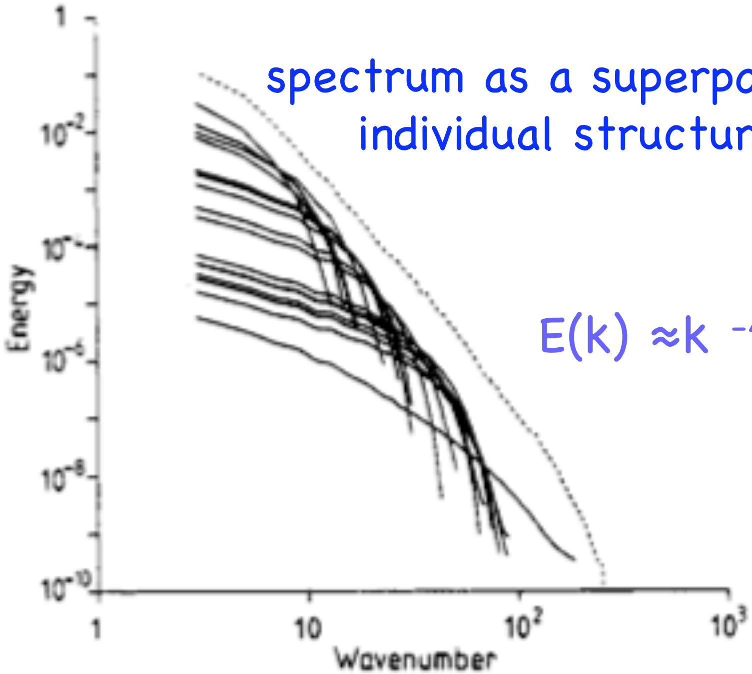
Steeper spectra ($k^{-\alpha}$ with $3 \leq \alpha < 5$) associate with the presence of strong large scale vortices, depending on forcing type.

Decaying turbulence shows long term memory of initial conditions (see McWilliams 1984)

**NO UNIVERSALITY**

long lived vortices






spectrum as a superposition of individual structures

$E(k) \approx k^{-4.3}$

Benzi, Paternello, Santangelo J. Phys. **A, 21** (1988)

# VIDEO

Thanks to S. Espa
Europhys. Lett. 71 (2005)

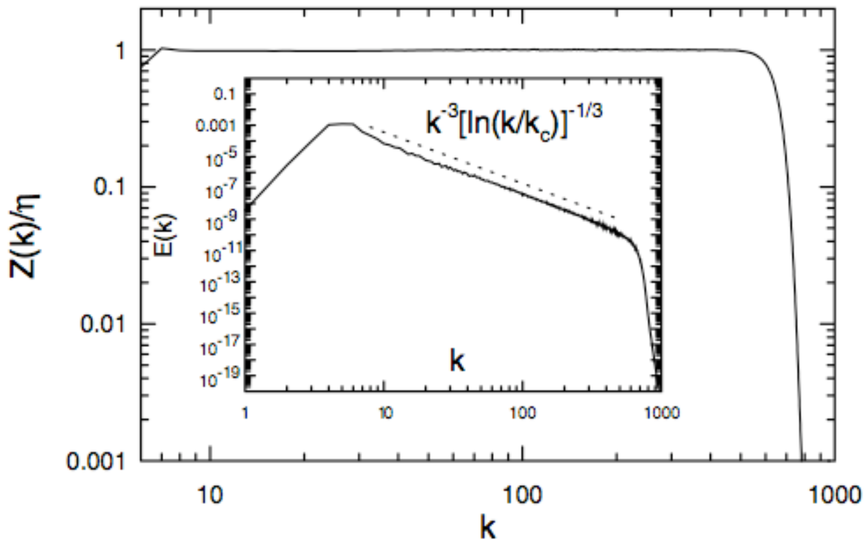
# More recently

Recent simulations of forced 2d direct cascade are more in agreement with Kraichnan prediction $E(k) \approx k^{-3}$ (+ logarithmic corrections)

V. Borue, PRL 71, (1993); S.Chen, R. Ecke, G. Eyink, X. Wang, Z. Xiao, PRL 91, (2003)
C.Pasquero, G.Falkovich, PRE 65 (2002); E.Lindborg, K.Alvelius, Phys. Fluids 12 (2000)

*This is always achieved if coherent structures are removed!*

$$\partial_t \omega + (u \cdot \nabla)\,\omega = \boxed{(-1)^{p_\alpha+1}\,\alpha\,\nabla^{-2\,p_\alpha}\omega} + \nu \nabla^2 \omega + F$$




Remark: $l_\nu = \left(\frac{\nu^3}{\zeta}\right)^{1/6}$

$$\boxed{N^2 = \left(\frac{L}{l_\nu}\right) \propto Re}$$

# Experimental direct cascade

NaCl solution stratified in density

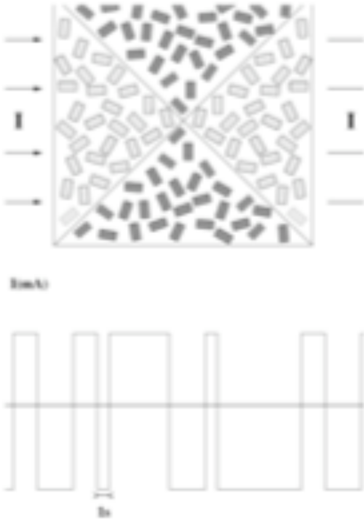
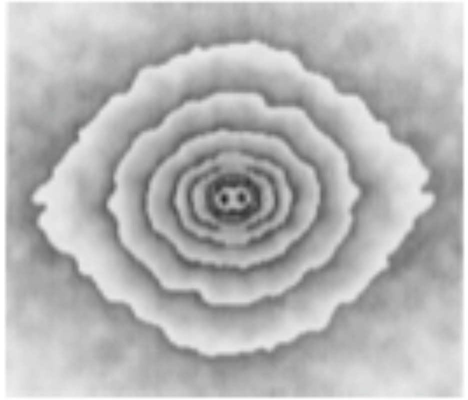



FIG. 1. A sketch of the arrangement of the magnets (as seen from above) and the time dependence of the electrical current crossing the cell. Black units have the same magnetic orientation; grey ones have the opposite one. The averaged lapse of time between two successive current switches is 2.5 s.



Flow is homogeneous, isotropic, stationary

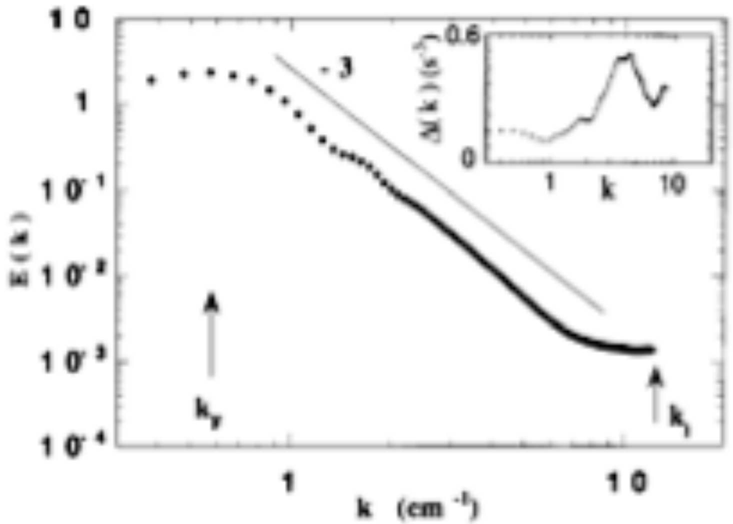


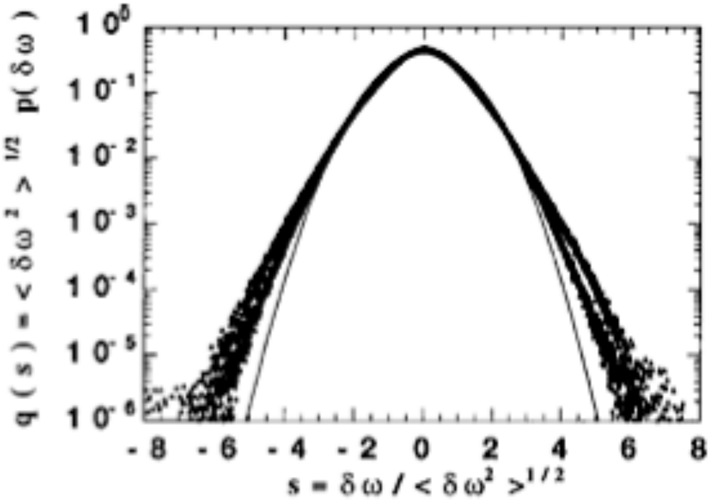

FIG. 3. Energy spectrum of the velocity field, averaged over 200 realizations of the velocity field in the statistically stationary state; the inset shows the enstrophy transfer rate $\Delta(k)$, calculated in similar experimental conditions.

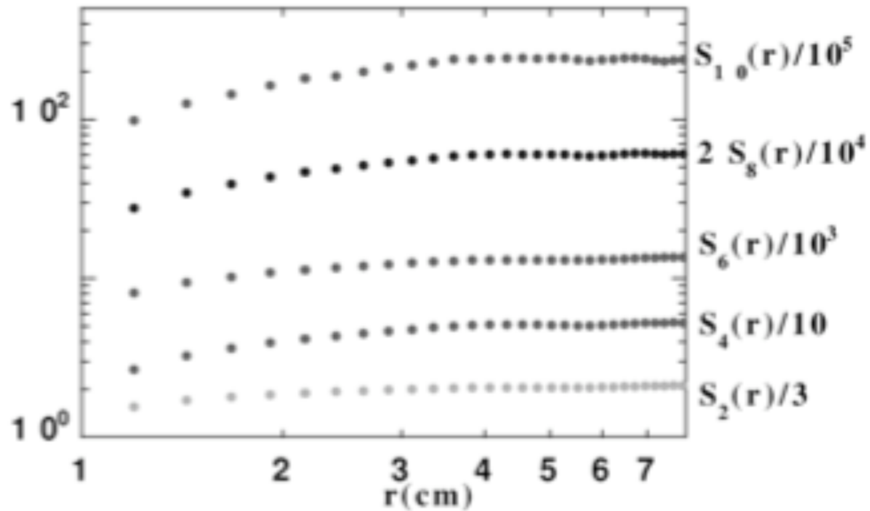
Jullien, Paret, Tabeling
Phys. Rev. Lett 83 (1999)

# Vorticity statistics in the direct cascade

• Vorticity statistics is non intermittent, in agreement with theoretical predictions *(Kraichnan - Batchelor, Falkovich & Lebedev 1994)*

$$S_p(r) = \langle [\omega(\mathbf{r}+\mathbf{r_0} - \omega(\mathbf{r_0})]^p \rangle = r^0 \mathcal{F}(l_F/|r|)$$



# Velocity statistics in the direct cascade

From the energy spectrum:

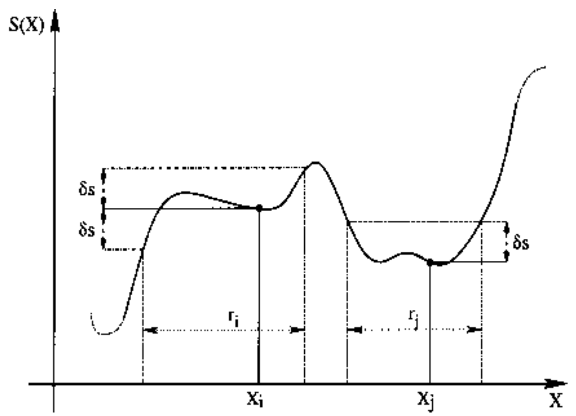
$$E(k) \sim k^{-3} ln^{-2/3}(k/k_F)$$

-> velocity field is smooth, $\delta u(r) \approx c_0\, r + c_1\, r^h$ with $h>1$,
this behaviour dominates standard structure functions at $r \ll l_F$:

$$S_p(r) = \langle [\mathbf{u}(\mathbf{r}+\mathbf{r_0} - \mathbf{u}(\mathbf{r_0}) \cdot \hat{\mathbf{r}}]^p \rangle \sim c_p\, r^p$$

Intermittency can possibly appear only in the scaling behaviour of the more-than-smooth fluctuations $r^h$ with $h>1$

--> *inverse structure functions*

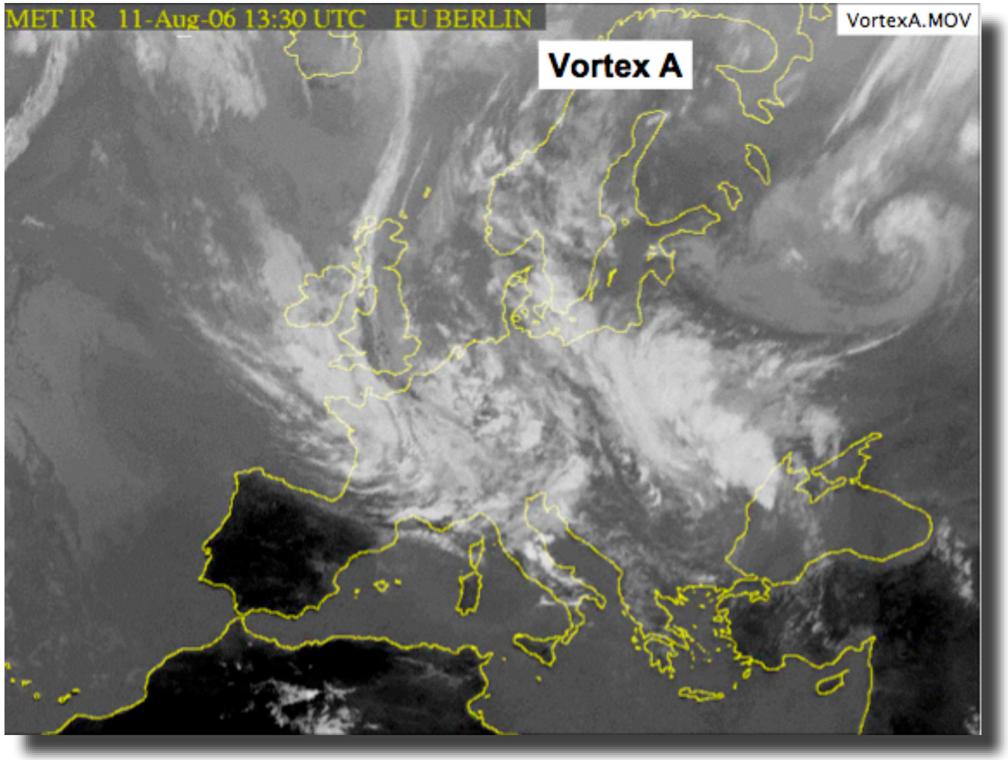



Biferale, Cencini, Lanotte,
Vergni, Vulpiani PRL 87 (2001)

# Part 4: an application of geophysical interest

- Predictability properties in the inverse energy cascade

*Boffetta & Musacchio Phys. Fluids* ***13****, (2001)*




*Early studies:*
*Leith 1971*
*Leith & Kraichnan, 1972*

# Predictability Problem

We deal with a generic dynamical system, knowing its evolution law and present status:

$$\begin{cases} \dfrac{dx}{dt} = F(x) \\ x(t_0) = x_0 \end{cases}$$

*Example:*
*weather forecasting*

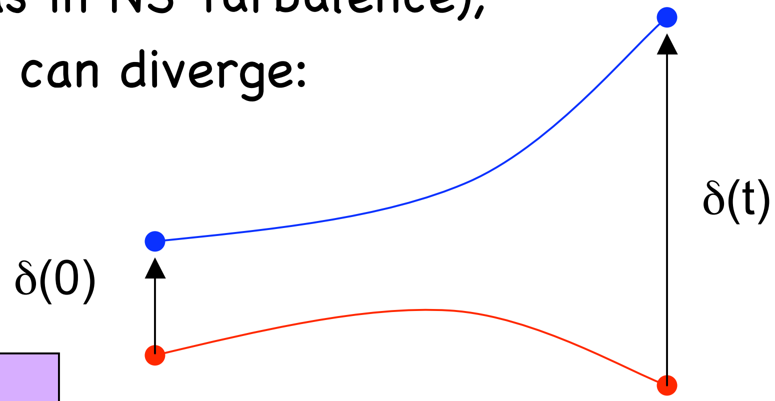
We want to know a future state, and the error associated.
In particular given a tolerance threshold $\delta_{max}$ about the future state:

**Predictability time $T_p$ is the maximum value at which one can forecast the system with a tolerance $\delta_{max}$**

# Dynamical Systems Tools

In chaotic dynamical systems (as well as in NS turbulence), two initially close states of the system can diverge:

$$\begin{cases} x_0, \dot{x}(t) = F(x, x_0) \\ x_1 = x_0 + \delta, \ \dot{x}(t) = F(x, x_1) \end{cases}$$




$$\boxed{|\delta(t)| \simeq |\delta(0)| e^{\lambda t}}$$

Infinitesimal perturbations grow exponentially with the maximum Lyapunov exponent:

$$\lambda \equiv \lim_{t \to \infty} \lim_{\delta(0) \to 0} \frac{1}{t} \, ln \frac{\delta(t)}{\delta(0)}$$

*Given tolerance Δ for the future state, and δ(0) as the initial error*

$$\boxed{T_P \equiv \frac{1}{\lambda} ln \frac{\Delta}{\delta(0)} \simeq \frac{1}{\lambda}}$$

- weak depend. on Δ,δ
- average time of system

# However….

In usual applications:

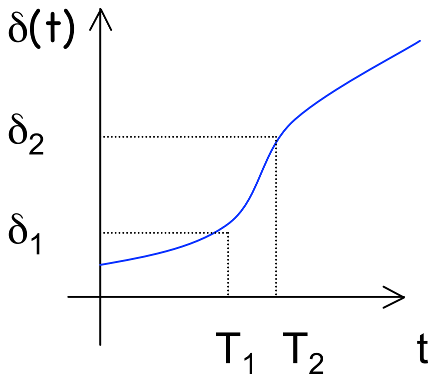
- *t* can be finite *(can not take limit t --> ∞)*
  or perturbation $\delta$ might be finite (*not infinitesimal)*

- perturbation $\delta$ can be on some degrees of freedom *(e.g. small scales)*,
  while we want to predict status of other degrees of freedom
  *(e.g. large scales)*

- details of the non-linear dynamics can be relevant
  ($T_p$ *independent of* $\lambda$, *but depends on* $\Delta,\delta$)

*We need a better estimate!*

# Finite Size Lyapunov Exponent *(FSLE)*




We define a series of thresholds $\delta_n=r^n\delta_0$, and we measure the time $T_r(\delta)$ such that $\delta(t+T_r)= r\,\delta(t)$

$$\lambda(\delta) \equiv \frac{1}{\langle T_r(\delta)\rangle} \ln r$$

***FSLE***

This represents the error growth rate at scale $\delta$

Linear limit (small perturbation)

$$\lambda \equiv \lim_{\delta\to 0} \lambda(\delta)$$

Time of error growth at scale $\delta$

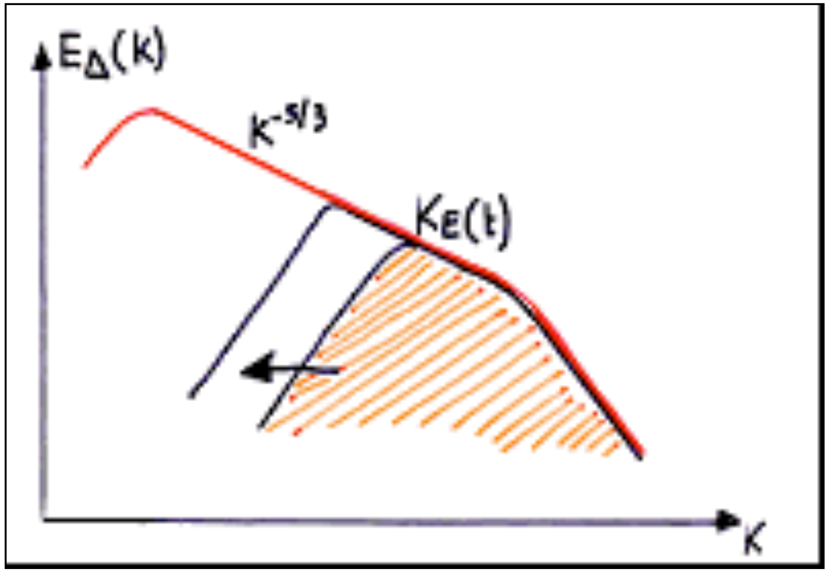
$$T_r(\delta) \simeq \frac{1}{\lambda(\delta)}$$

*Crisanti, Jensen, Paladin, Vulpiani PRL 70 (1993)*

FSLE APPLICATIONS:
Predictability problems; Relative dispersion of Lagrangian particles;
Experimental data analysis: laboratory experiments, ocean drifters;....

# Predictability Problem in Turbulence

In the inverse cascade the initial infinitesimal error at scale $k_E(t=0)$ gets to larger and larger scales




how does the error spectrum grows when the error is at wavenumber in the inertial range?

# How can we model this?

*Transfer time at scale k ≈ τ(k) local turn-over time*

From Kolmogorov scaling: -->the scale attained by the error at time t:

$$\begin{cases} E(k) \sim \epsilon^{2/3} k^{-5/3} \\ \tau(k) \sim \epsilon^{-1/3} k^{-2/3} \end{cases} \Rightarrow \boxed{k_E(t) \sim \epsilon^{-1/2} t^{-3/2}}$$

The error spectrum is:

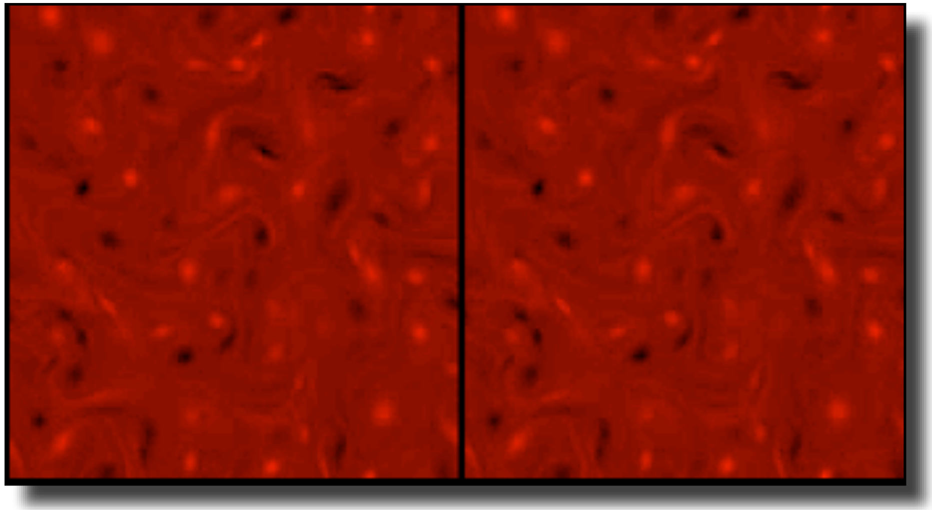
$$E_\Delta(k,t) = \begin{cases} E(k) & k > k_E(t) \\ 0 & k < k_E(t) \end{cases} \Rightarrow \boxed{E_\Delta(t) \equiv \int_0^\infty E_\Delta(k,t)dk = G\epsilon t}$$

*Local error grows algebraically (not exponentially)*
*Global error energy grows diffusively*
*Large scale predict. independent of Lyapunov*

*G=4.19*
Kraichnan, Leith 1972

# DNS of inverse cascade

u-field



perturbed u′

We consider a field u(x,t) and a slightly perturbed field at small scales u′(x,t) :

$$\delta(x,0)=|\, u(x,0) - u'(x,0)| << 1$$

and we integrate their dynamics numerically (*usual set up*)

*At later times, we want to estimate:*

$$\delta(\mathbf{x},t) = |\mathbf{u}(\mathbf{x},t) - \mathbf{u}'(\mathbf{x},t)|$$

# Predictability in 2d inverse cascade

*Given the error of amlplitude:*

$$\delta(\mathbf{x},t) = |\mathbf{u}(\mathbf{x},t) - \mathbf{u}'(\mathbf{x},t)|$$

If error amplitude smaller than fluctuations at the forcing scale $\delta < \delta u(\eta)$ :

$$\lambda(\delta) = \lambda \sim \tau^{-1}(\eta)$$

If error amplitude are fluctuations of the inertial range: $\delta u(\eta) < \delta < \delta u(L)$ :

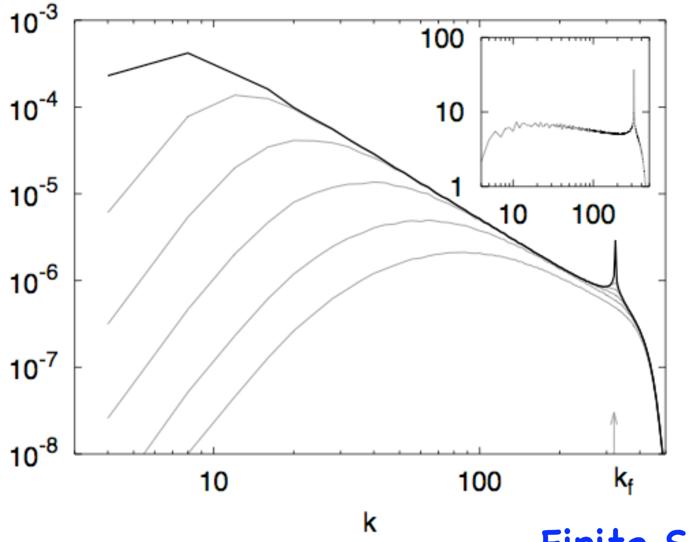
$$\lambda(\delta) \approx \tau^{-1}(\delta u(l) = \delta) \approx \epsilon^{1/3} l^{-2/3} \approx \epsilon \delta^{-2}$$

$$\lambda(\delta) = \begin{cases} \lambda & \text{infinitesimal} \\ A\epsilon\delta^{-2} & \text{inertial range} \\ 0 & \text{saturation at large scale} \end{cases}$$

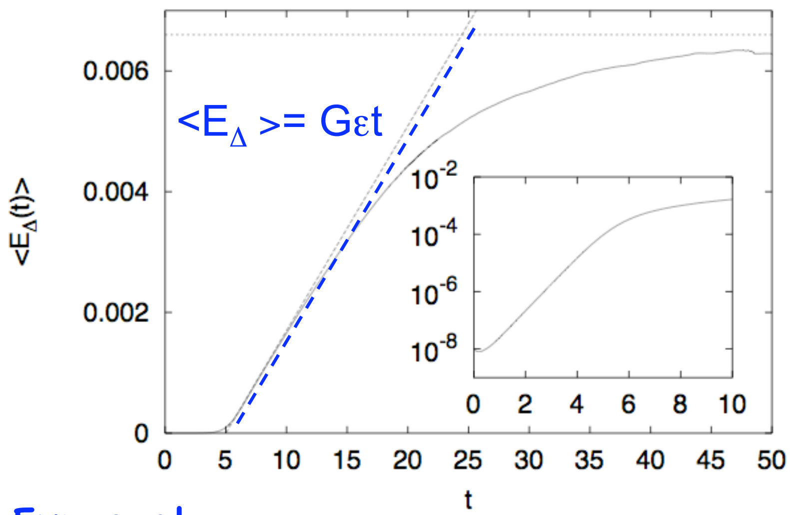
*FSLE for turbulence*

# Results from DNS of 2d inverse cascade

## Self-similar growth of error

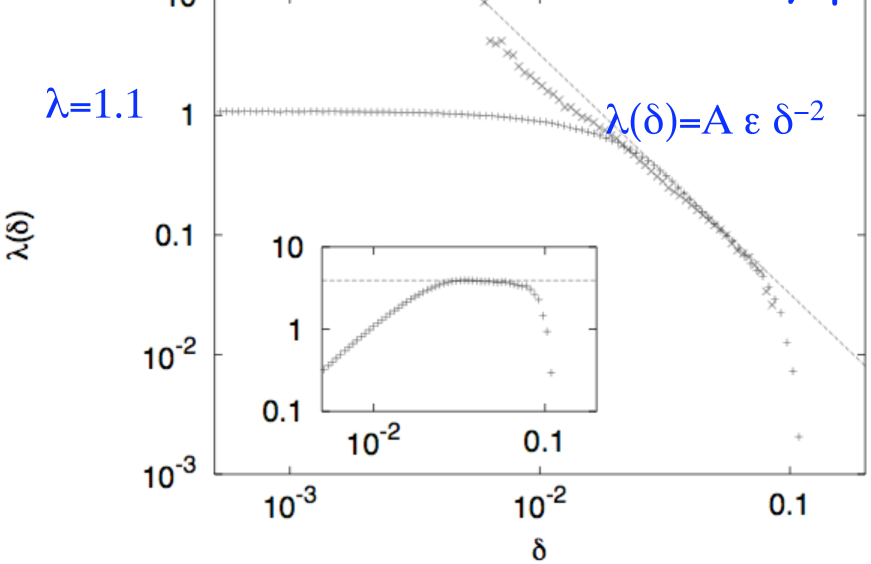



## Average error energy growth




G=4.1 ± 0.1

## Finite Size Lyapunov Exponent




A=3.9

**Chaotic dynamics creates uncorrelated energy 4 times faster than energy transfer**

G is in very good agreement with theoretical predictions of Kraichnan & Leith with *Test Field Model*

*Boffetta & Musacchio Phys. Fluids* ***13****, (2001)*

# Predictability Time

For error amplitude in the inertial range: $\lambda(\delta) = A\,\varepsilon\,\delta^{-2}$

We can calculate the predictability time associated to a maximum tolerance $\Delta$ :

$$T_p = \int_{\delta_0}^{\Delta} \frac{d\delta}{\delta\lambda(\delta)} = \frac{1}{2A\epsilon}(\Delta^2 - \delta_0^2) \sim \frac{1}{2A\epsilon}\Delta^2$$

In terms of the error spectrum, by dimensional analysis

$$\Delta^2 = 2E_\Delta = 3C^{3/2}\epsilon\tau_k \Rightarrow \boxed{T_p(\Delta) = \frac{3C^{3/2}}{A}\tau_k \simeq 5.6\tau_k}$$

*Predictability time for an error at scale k*

*Example:*
*In the stratosphere L=500km, $\tau(L)$=1 day :* $T_p \approx 6$ *days*

# *Final part*

- Uncovered and/or open issues
- New approaches to 2d turbulence

# A brief tour in the "untold"

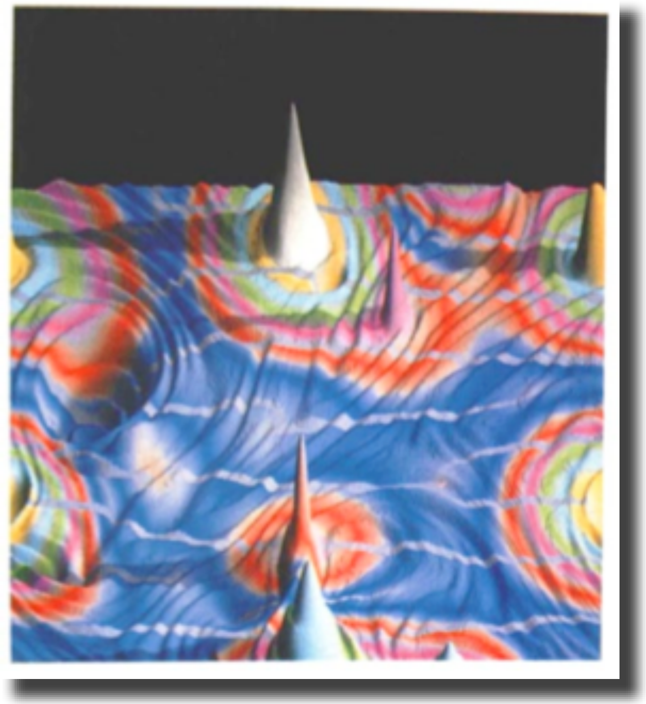
**Some of the uncovered issues:**

- Direct cascade with linear friction "$-\alpha\ \omega$"
  *(analogy with passive scalar transport at small scales)*

  (remember Massimo Cencini talk)

- Coherent structures in an un-coherent background

- Wavelet approaches to 2d turbulence

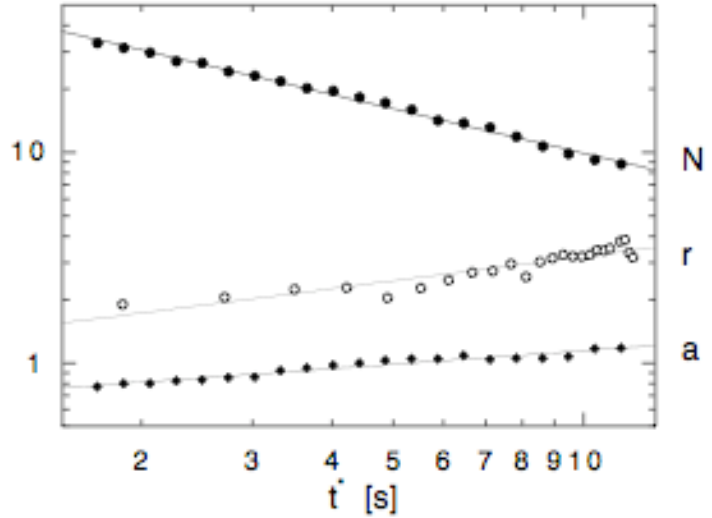


Farge, Ann. Rev Fluid Mech 24 (1992)

## Some of the uncovered issues:

- Statistical mechanical theories (*equilibrium, conservative*) for 2d turbulent flows

- Decaying 2d turbulence : *(much different from steady case)* from Batchelor's self-similar theory to more recent observations




N=Number density

r=intervotex distance

a=size

# A brief tour in the unknown (III)

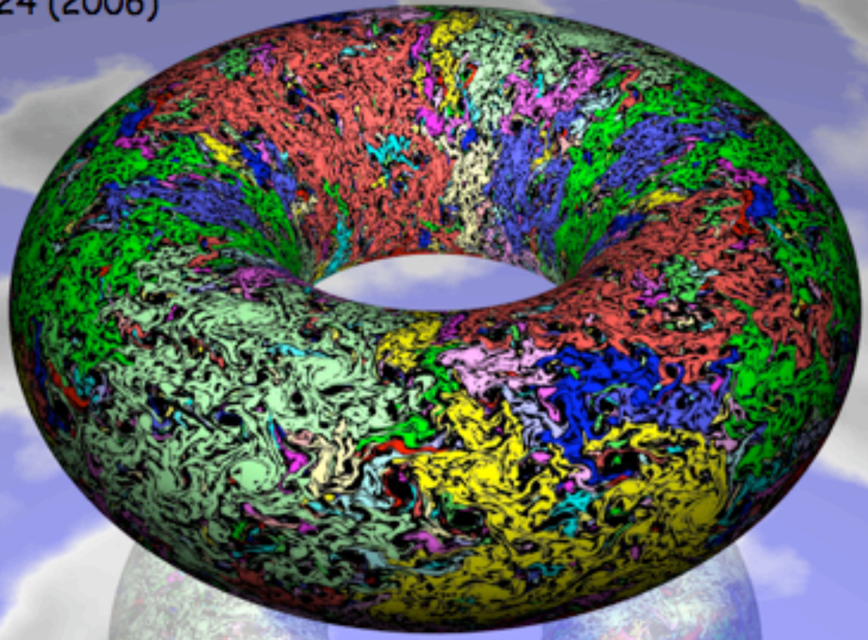
**Some of the open issues:**

- A well established theory for decaying 2d turbulence
- Theoretical understanding of the direct cascade energy spectrum shape
- Universality issues?
- Satisfactory description of coherent structures, of their role, and link to statistical theories
- Isotropy restoration in 2d flows?
- A Lagrangian understanding of 2d turbulence:
  - *partial for direct cascade*
  - *absent for inverse cascade*

# Conformal invariance in 2d inverse cascade

D.Bernard, G.Boffetta,
A.Celani and G.Falkovich
Nature Physics, **2** 124 (2006)

www.ph.unito.it/~boffetta

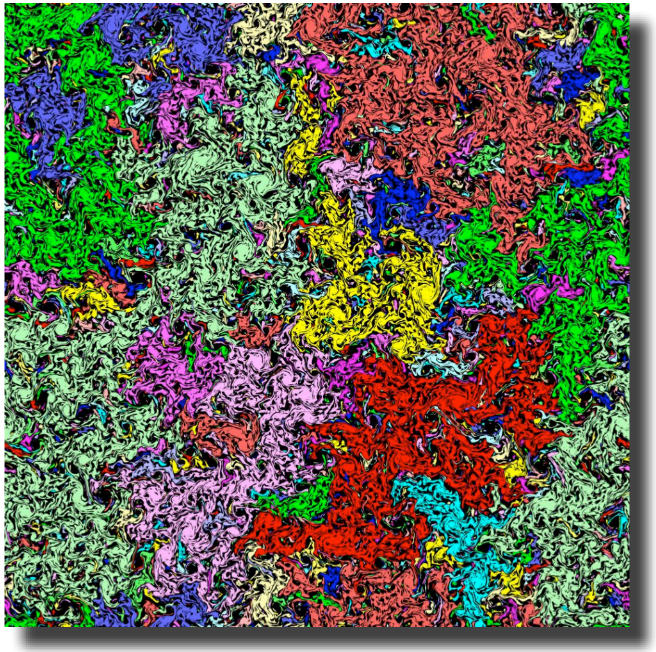


Statistical mechanics of two-dimensional turbulence

Zero-vorticity isoline are conformally invariant random curves
They are compatible with $SLE_6$

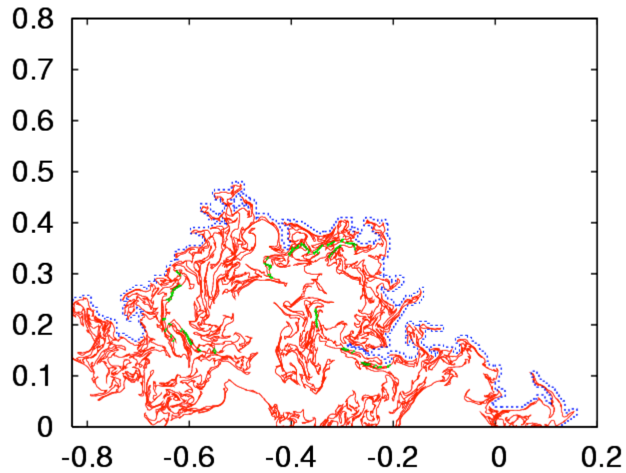
Beyond geometry: conformal invariance for correlation functions ?
What about other 2D turbulent systems ?

# Conformal invariant observables

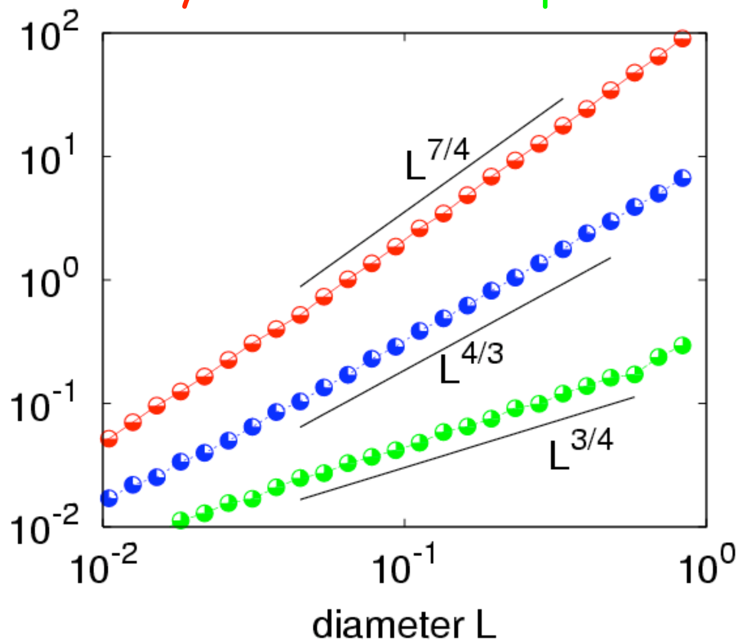


Vorticity clusters characterized with fractal dimensions as in critical system

*This result is no longer true if phases are randomized !*




Boundary; Frontier; Cut points



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Thanks for your attention