Small-scale turbulence; theory, phenomenology and applications

Stretched vortices as basis for SGS modeling

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Overview

- Motivation; why LES?

- Expectations of LES. Some present models.

- Stretched-vortex sub-grid scale model
  - *Structure-based SGS model (2C-3P)*
  - *SGS stresses*
  - *SGS vortex orientations*
- Example Applications
  - *Decaying incompressible turbulence*
  - *Channel flow at moderate Re_\tau*
  - *Aircraft trailing vortices*

- Subgrid-flux model for passive scalar
  - *Overholt-Pope test*

- LES of compressible turbulence

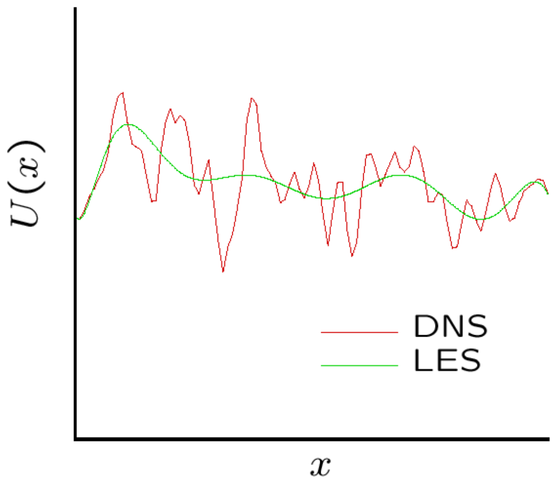
Why LES?

Table 2. Characteristics of some representative channel-flow simulations.

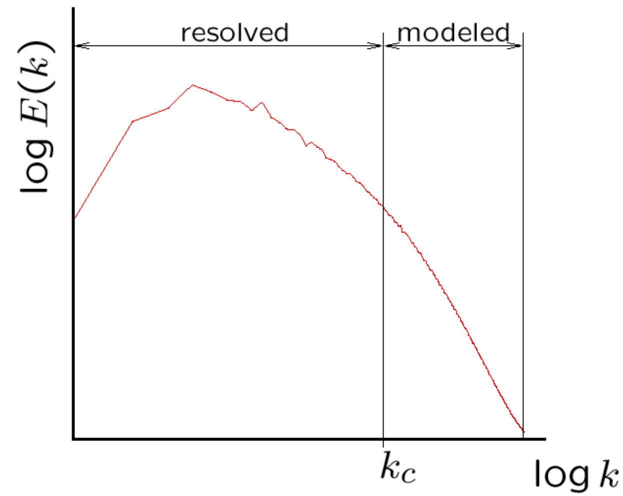
Re_τ	L_x/δ	L_z/δ	Points	Year
180	12	6	5 M	1987 [11]
590	6	3	40 M	1997 [12]
550	25	12	600 M	2001 [13]
950	25	9	4 G	2003 [14]
1 900	3	1.5	450 M	2003 [14]
10 000	12	6	900 T	2015?

- Jiménez (JOT, 2003)

Large-Eddy Simulation



physical space: fine-scale fluctuations not resolved, their influence is modeled.



spectral space: resolved range, $k < k_c$ (cutoff wavenumber k_c), subgrid range $k > k_c$.

LES and SGS modeling

- LES equations

$$\frac{\partial \tilde{U}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\tilde{U}_i \tilde{U}_j\right) = -\frac{\partial \tilde{P}}{\partial x_i} - \frac{\partial T_{ij}}{\partial x_j} + \nu \frac{\partial^2 \tilde{U}_i}{\partial x_j \partial x_j} + F_i$$

Large-eddy simulation (LES) makes modeling assumptions;

- $T_{ij} = \widetilde{U_i U_j} - \tilde{U}_i \tilde{U}_j$; Subgrid stresses are replaced by some model T_{ij}: subgrid stress (SGS) model:

$T_{ij} = T_{ij}[\widehat{U}_j, \partial \widehat{U}_i / \partial x_j, ...]$

- $\tilde{U}_j \rightarrow \widehat{U}_j$; Filtered field $\tilde{U}_j$ is modeled by a computed under-resolved field $\widehat{U}_j$.

What can (should) we expect from LES?

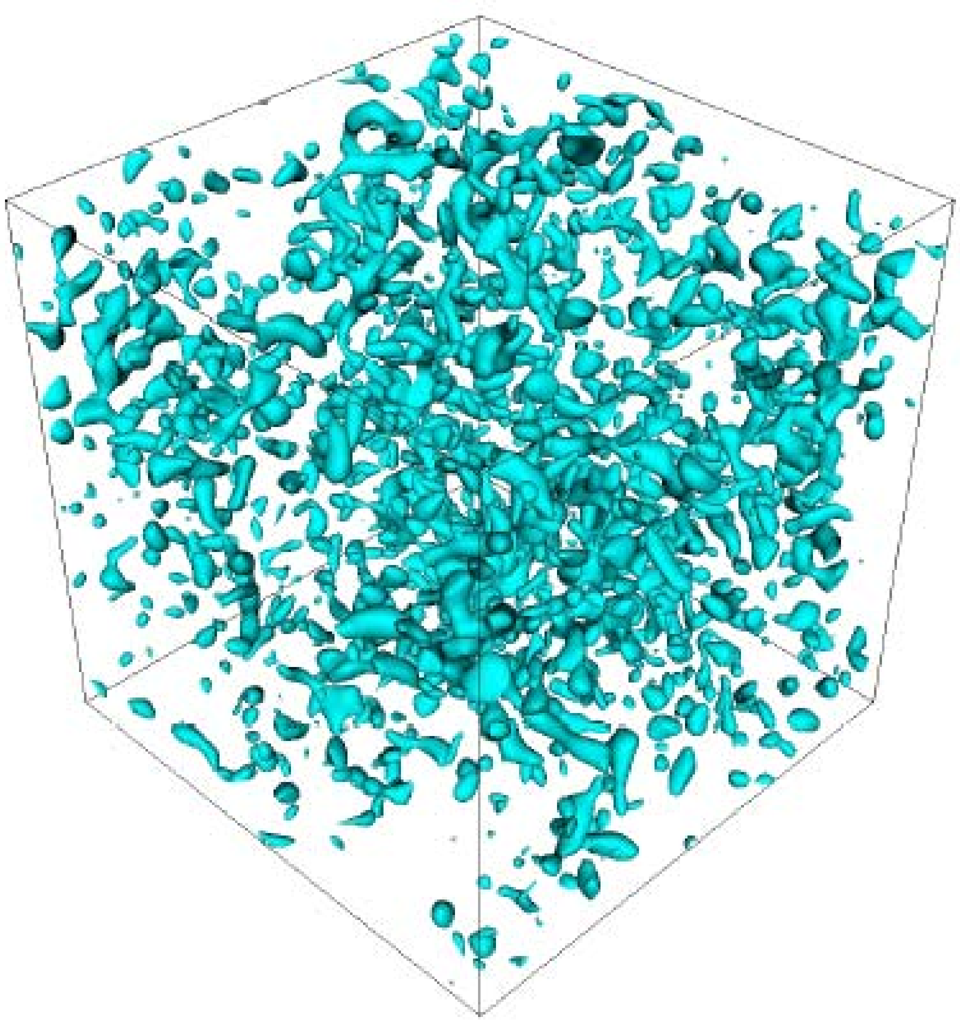
- Robustness for different flows at large Re
- One-point statistics (velocity, density, concentration)
- Two-point statistics across full wavenumber range?
- Predictive for turbulent mixing
- Estimates for full turbulent fields
  - *Not just the 'filtered' part*
  - *Multiscale LES*
- Knowledge of Reynolds number; what is it?
- Fast convergence to DNS
  - *In some cases DNS is not available*
  - *DNS not achieved for compressible turbulence containing strong shocks*

SGS models

- Smagorinsky
- Dynamic Smagorinsky (CTR, Germano)
- Eddy-viscosity structure function (Lesieur, Mètais)
- Scale-similarity (Bardina)
- MILES (Boris,Grinstein)
- Approximate deconvolution model (Leonard, Adams)
- Optimal LES (Adrian, Moser)
- Stretched-vortex subgrid model

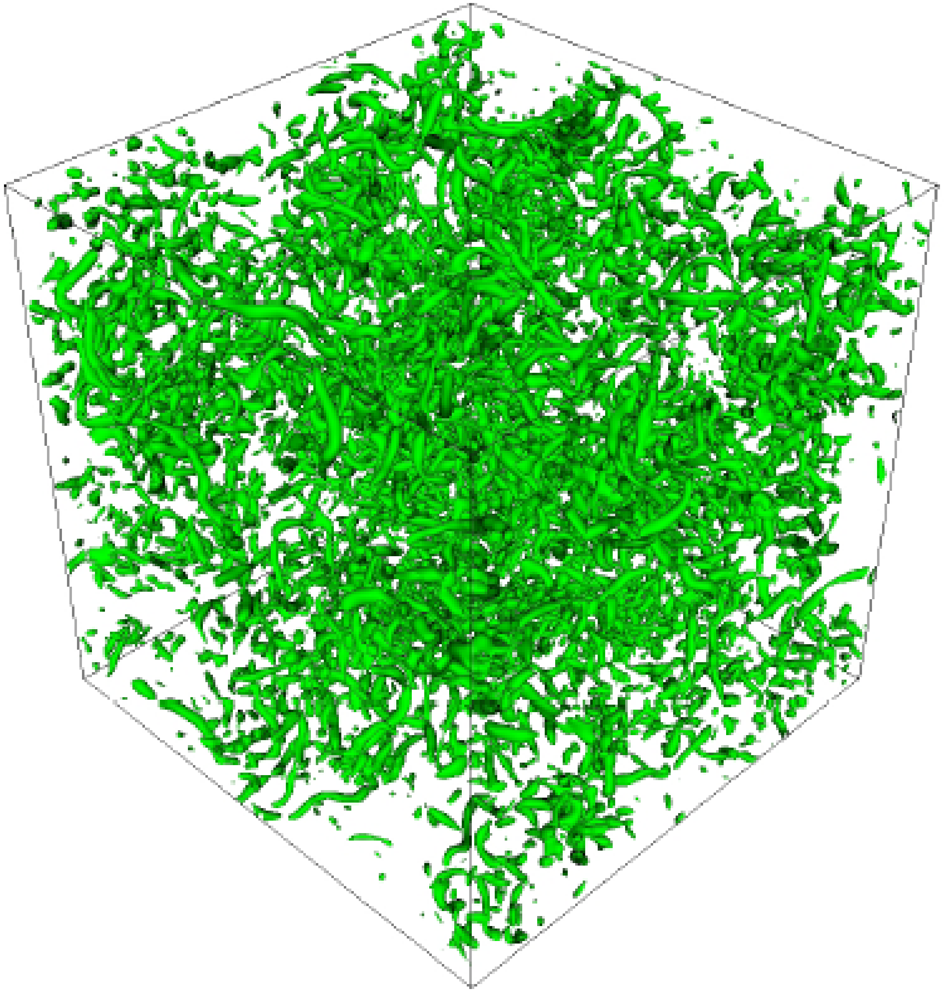
Structure of vorticity: curvelet transform

- 512^3, k_max*η = 2.08 (Horiuti)
- R_λ = 77
- Curvelet transform; scale 2



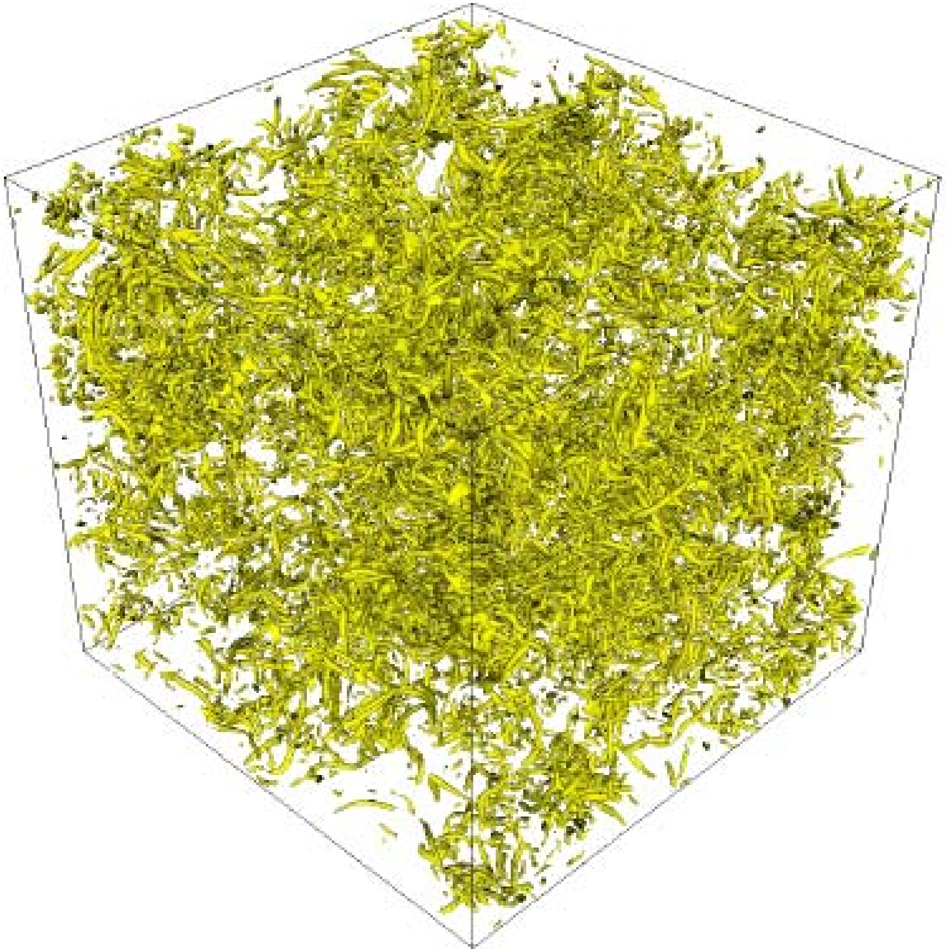
Structure of vorticity: curvelet transform

- 512^3, $k_{max}*\eta = 2.08$
- $R_\lambda = 77$
- Curvelet transform; scale 3



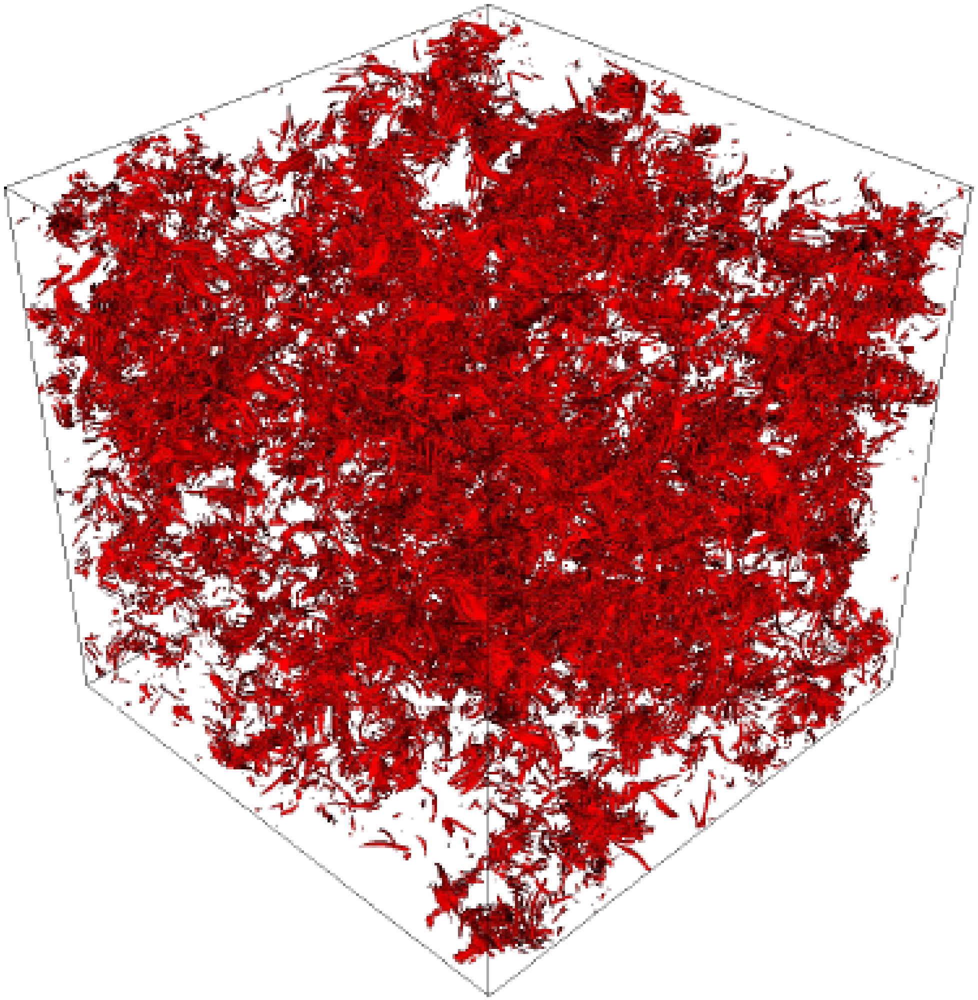
Structure of vorticity: curvelet transform

- 512^3, k_max*η = 2.08
- R_λ = 77
- Curvelet transform; scale 4

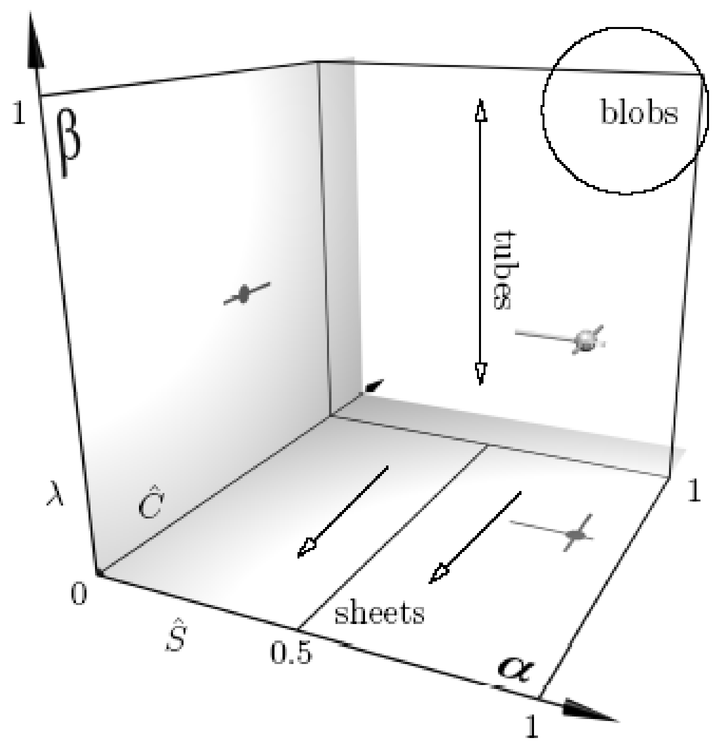


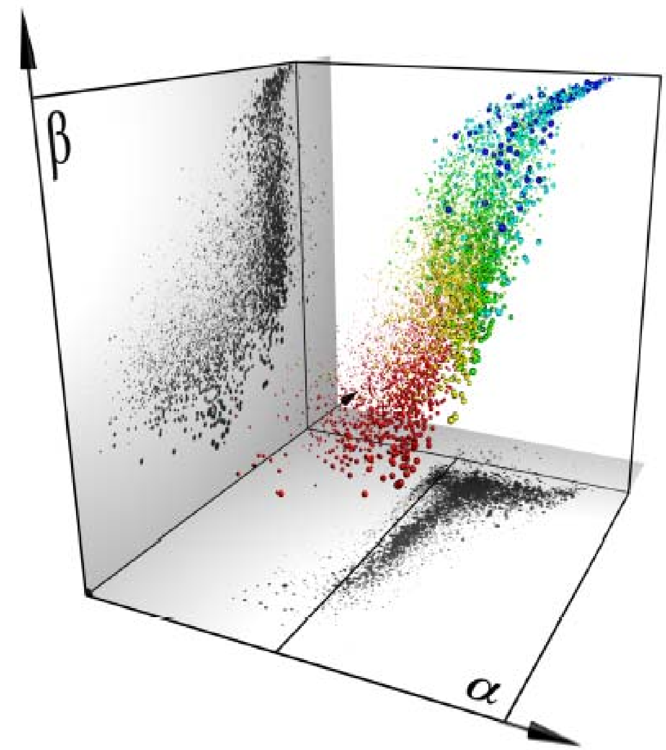
Structure of vorticity: curvelet transform

- 512^3, k_max*η = 2.08
- R_λ = 77
- Curvelet transform; scale 5



Structure of vorticity: curvelet transform

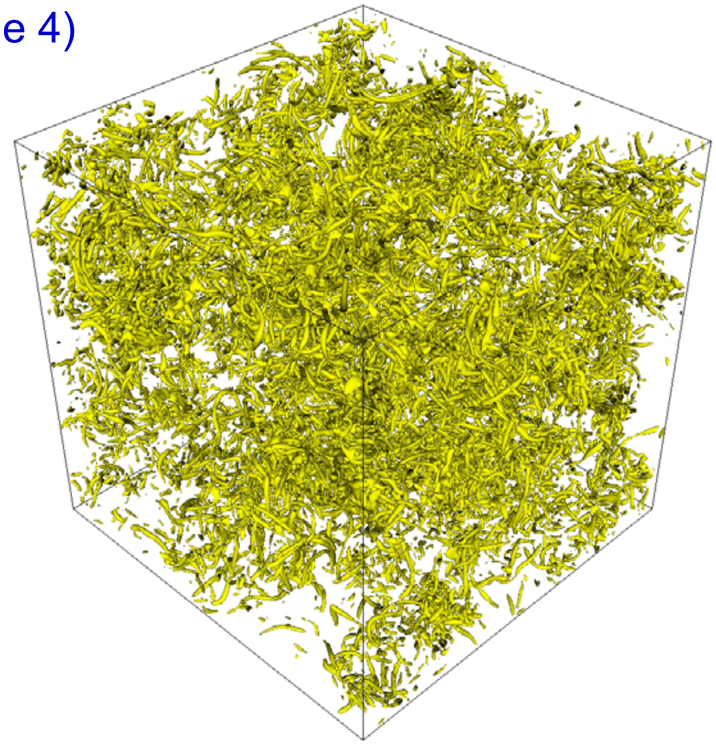


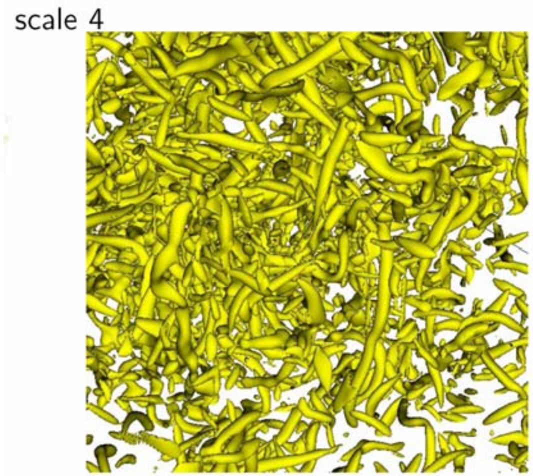


512^3 - scales 1-5

Explicit SGS model; stretched-vortex model

512^3 DNS (scale 4)



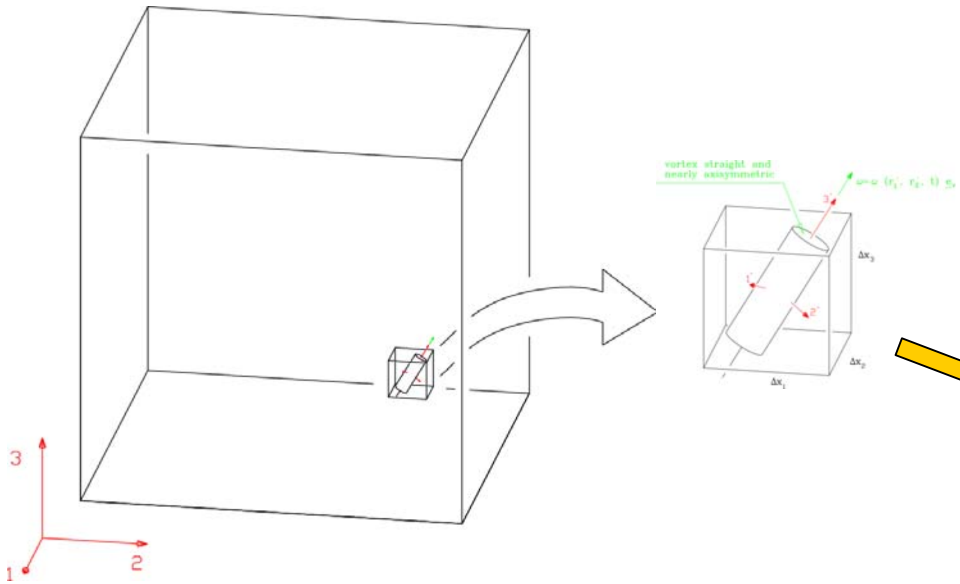


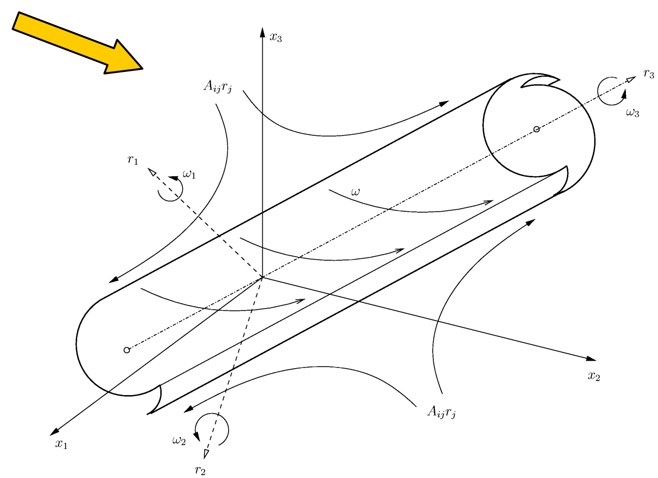
- Small scales of turbulence Intense vorticity in form of ``worms''

  Ashurst, Jimenez et al (1993)

- Can this be used as a basis of a stucture-based sub-grid scale model?

Subgrid-scale vortex: 2C-3P model





- Structure-based approach
- Subgrid motion represented by nearly axisymmetric vortex tube within each cell
- Vortex tube modeled by columnar vortex embedded in time-varying linear field
- Time-varying linear field is provided by local velocity-gradient of resolved-scale flow

Explicit SGS model; stretched-vortex model

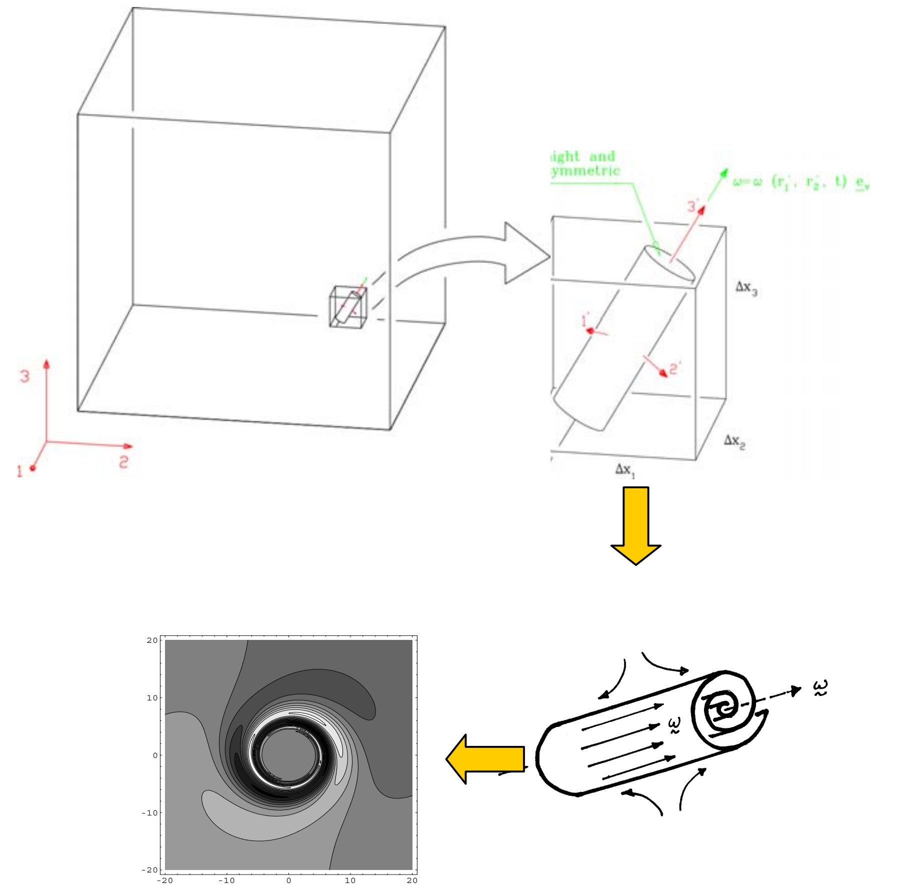
- Structure-based approach
- Subgrid motion represented by nearly axisymmetric vortex tube within each cell
- Local solution of NS equations for stretched-spiral vortex
  - *Lundgren (1982), Pullin & Lundgren (2001)*
- Subgrid transport:

$$\tau_{ij} = \overline{\rho}\tilde{K}(\delta_{ij} - e_i^v e_j^v)$$

$$q_i^T = -\overline{\rho}\frac{\triangle_c}{2}\tilde{K}^{1/2}(\delta_{ij} - e_i^v e_j^v)\frac{\partial(\tilde{c}_p\tilde{T})}{\partial x_j}$$

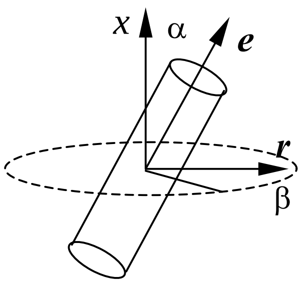
$$q_i^\psi = -\overline{\rho}\frac{\triangle_c}{2}\tilde{K}^{1/2}(\delta_{ij} - e_i^v e_j^v)\frac{\partial\tilde{\psi}}{\partial x_j}$$

$$\tilde{K} = \int_{k_c}^{\infty} E(k)dk, \qquad k_c = \pi/\Delta_c$$



Model parameters

- Subgrid energy spectrum (Lundgren, 1982)



$$
\begin{aligned}
E(k) &= \mathcal{K}_0 \epsilon^{2/3} k^{-5/3} \exp[-2k^2 \nu / (3|\tilde{a}|)] \\
\tilde{a} &= \tilde{S}_{ij} e_i^v e_j^v, \qquad \tilde{S}_{ij} = \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right)
\end{aligned}
$$

- Parameters obtained from resolved-scale, second order velocity structure-functions (Lesieur et al)
- Spherically averaged structure functions

$$
\begin{aligned}
\mathcal{K}_0 \epsilon^{2/3} &= \frac{\overline{\mathcal{F}_2}(\triangle)}{\triangle^{2/3} \mathrm{A}}, \qquad \mathrm{A} = 4 \int_0^{\pi} s^{-5/3}(1 - s^{-1} \sin s) ds \approx 1.90695 \\
\overline{\mathcal{F}_2}(\triangle) &= \frac{1}{6} \sum_{j=1}^{3} \left(\delta \tilde{u}_1^{+\,2} + \delta \tilde{u}_2^{+\,2} + \delta \tilde{u}_3^{+\,2} + \delta \tilde{u}_1^{-\,2} + \delta \tilde{u}_2^{-\,2} + \delta \tilde{u}_3^{-\,2} \right)_j ,
\end{aligned}
$$

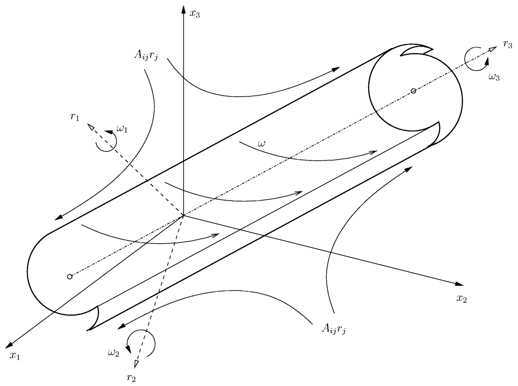
Subgrid vortex orientation ***e***

- Phenomenology
  - λ *: fraction aligned with principal extensional eigenvector of resolved rate-of-strain tensor* (*corresponding eigenvalue,* λ_3)
  - (1- λ) *: fraction aligned with resolved vorticity vector,* ω (*Misra & Pullin 1997*)

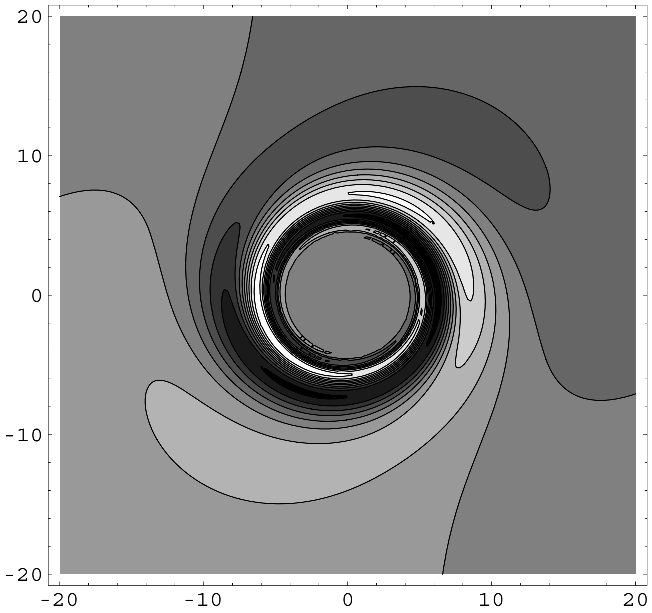
$$\lambda = \frac{\lambda_3}{\lambda_3 + \|\boldsymbol{\omega}\|}$$

- 2C-3P dynamic rotation model
  - *Subgrid vortex orientation responds to local time-varying resolved-scale rate of strain tensor A_ij*

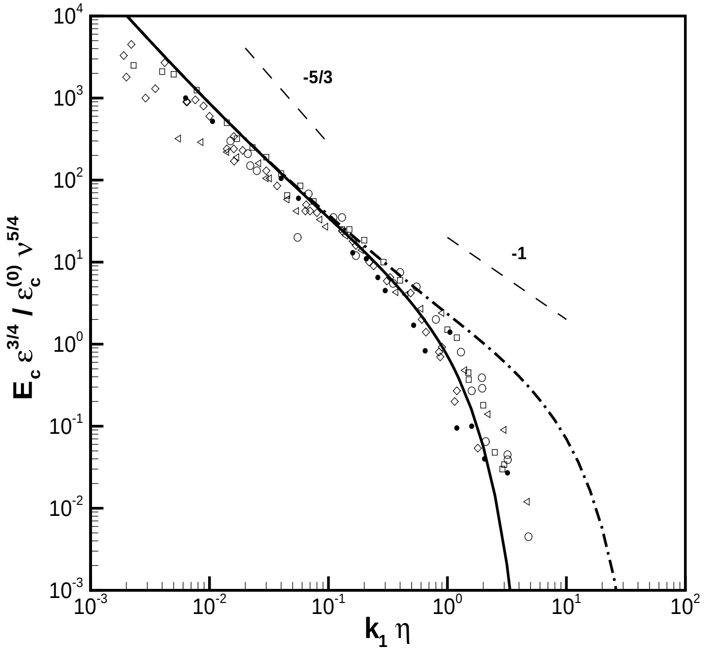
$$\frac{\partial e_i}{\partial t} = e_j \widetilde{A}_{ij} - e_i e_k e_j \widetilde{A}_{kj}$$



Scalar spectrum from stretched-spiral vortex



Schematic showing winding of scalar field by `subgrid vortex'. Contours of passive scalar



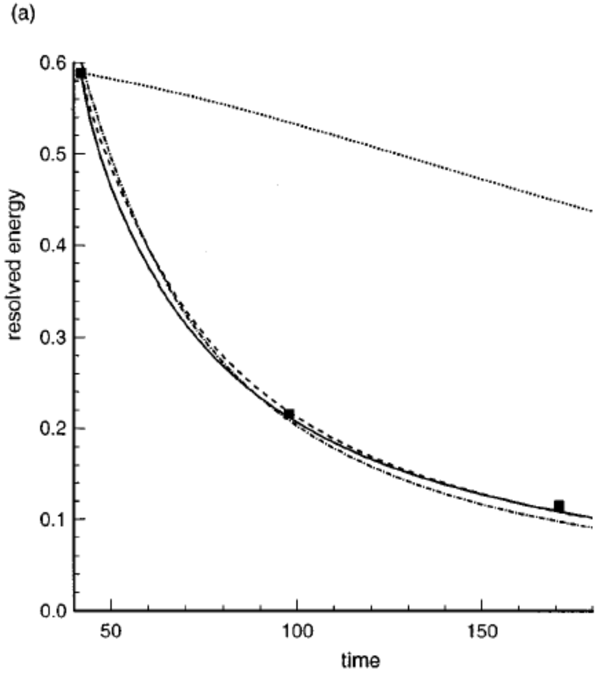
1-D scalar spectrum for homogeneous turbulence,Pullin & Lundgren (2001)

______ Sc = 7, ------- Sc = 700. Symbols, Data

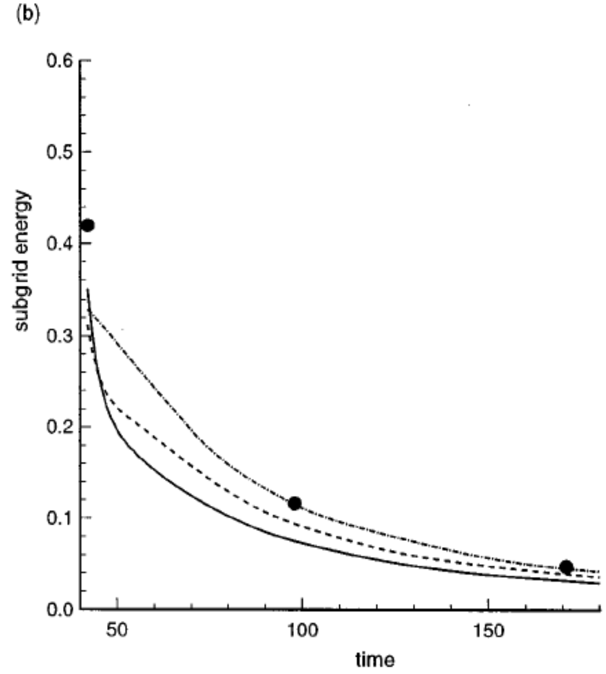
(Gibson & Schwarz 1963)

Decay of homogeneous turbulence

- 32^3 LES of decaying turbulence; R_\lambda = 70 (PF, 1997)
- Data; Comte-Bellot & Corssin (1971)



Resolved-scale energy

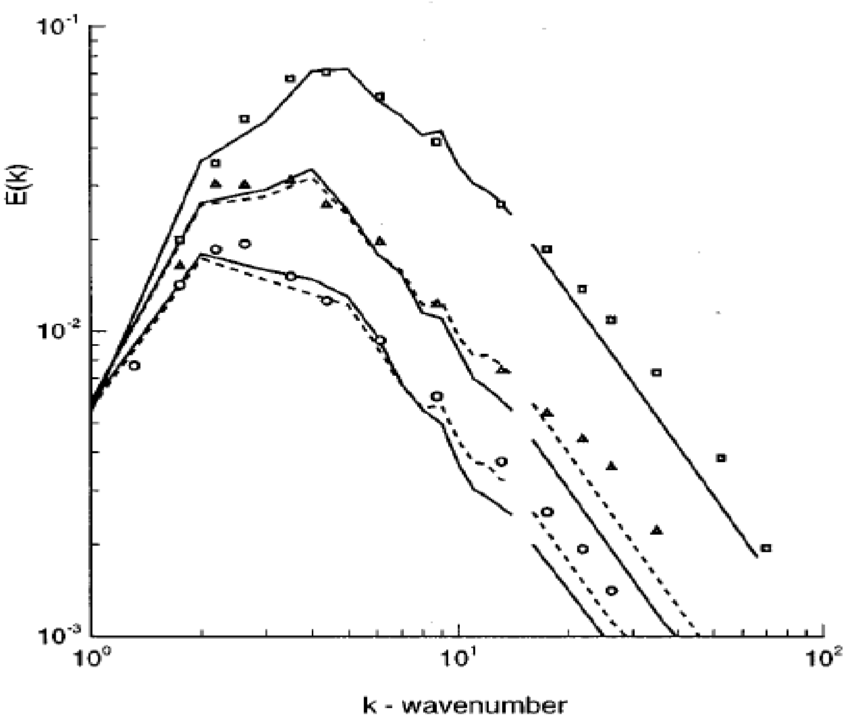


Subgrid-scale-scale energy

Decay of homogeneous turbulence

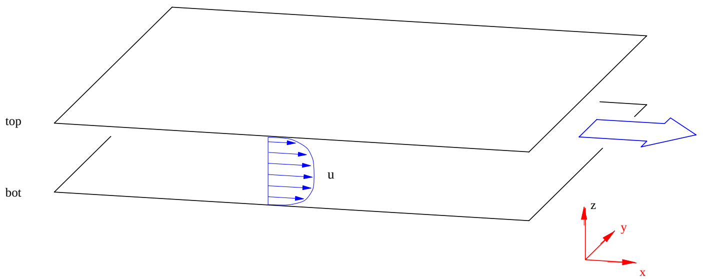
- 32^3 LES of decaying turbulence; R_\lambda = 70 (PF, 1997)
- Data; Comte-Bellot & Corssin (1971)

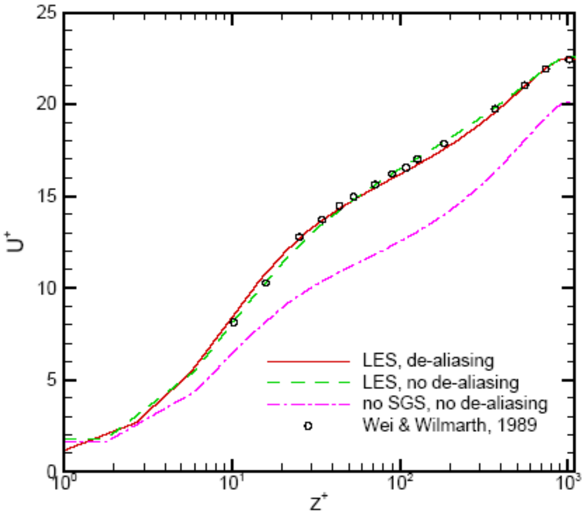
(a)

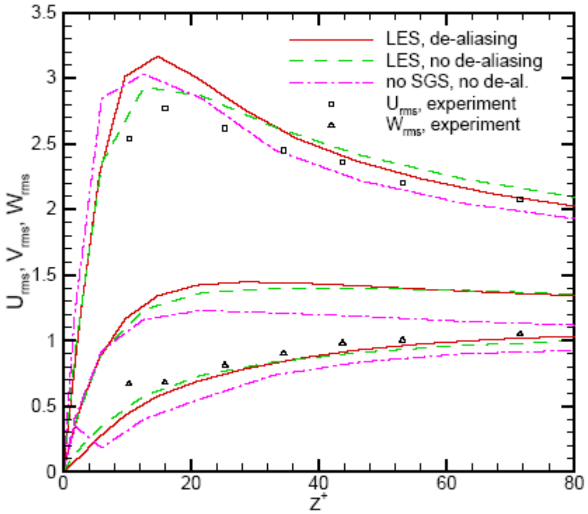


Velocity (energy) spectrum

LES of turbulent channel flow: Re_tau = 1017





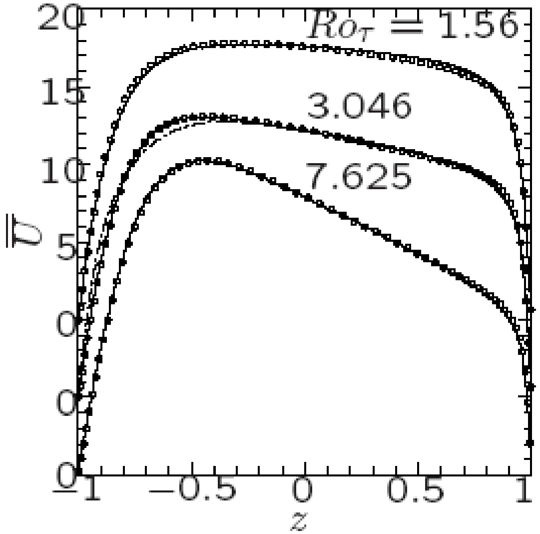


Domain size: $2.5\pi\delta \times \pi\delta \times 2\delta$,

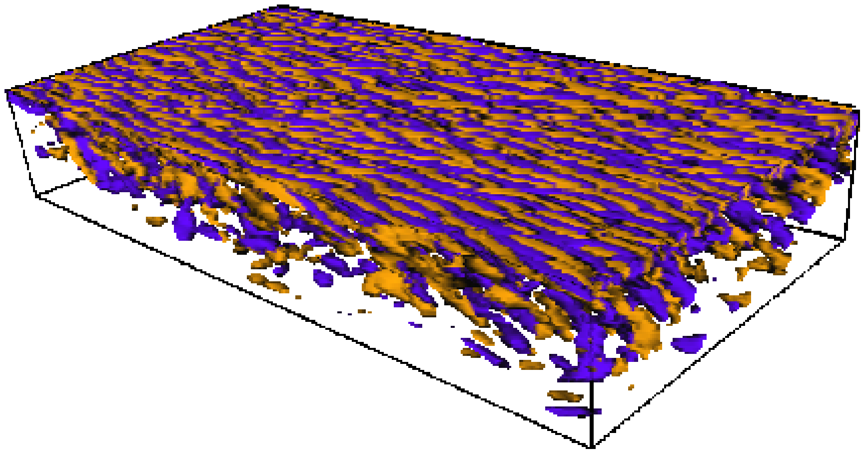
effective resolution: $48 \times 64 \times 65$.

Partial Resolution of viscous sublayer !!!!!!

Rotating channel flow

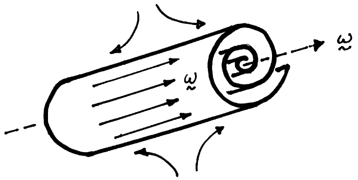


Resolution: $48 \times 64 \times 43$ (DNS: $128 \times 128 \times 128$)



$Ro_\tau = 7.625$, $\omega_z = \pm 15 u_\tau / h$

SGS model for subgrid flux of a passive scalar



- Filtered equation for a passive scalar ϕ is

$$\frac{\partial \tilde{\phi}}{\partial t} + \frac{\partial}{\partial x_j}\left(\tilde{\phi}\tilde{U}_j\right) = -\frac{\partial g_j}{\partial x_j} + D\frac{\partial^2 \tilde{\phi}}{\partial x_j \partial x_j}$$

$$g_j = \widetilde{\phi U_j} - \tilde{\phi}\tilde{U}_j$$

g_j is the subgrid flux of ϕ by the turbulent velocity field.

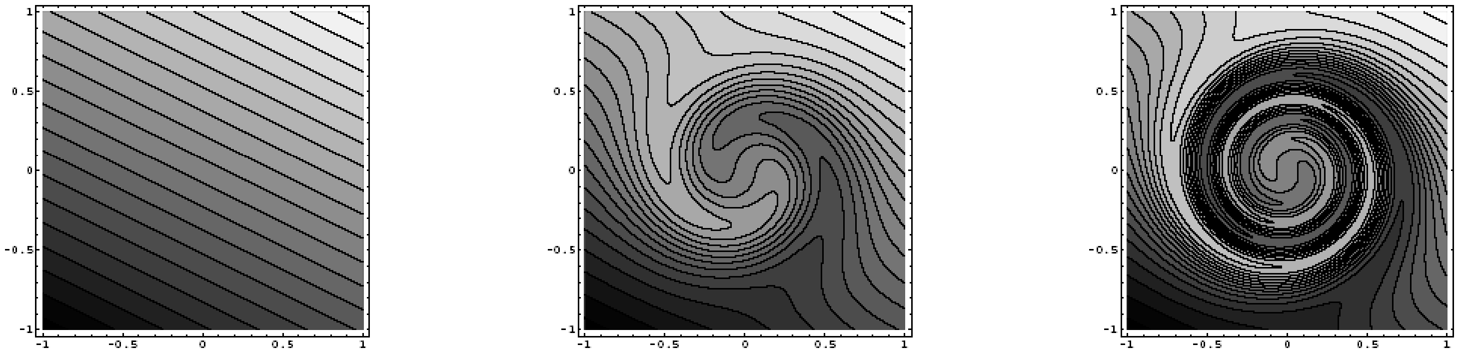
- We model g_j by the winding of $\tilde{\phi}$ field by an *axisymmetric* model subgrid vortex

SGS model for subgrid flux of a passive scalar

- Subgrid velocity field. Scalar convection equation

$$u_\theta = r\,\Omega(r), \qquad u_r = u_{x'_3} = 0, \qquad \frac{\partial \phi}{\partial t} + \Omega(r)\,\frac{\partial \phi}{\partial \theta} = 0$$

$$\phi(r,\theta,t) = r\cos[\theta - \Omega\,t]\left(\frac{\partial \tilde{\phi}}{\partial x'_1}\right) + r\sin[\theta - \Omega\,t]\left(\frac{\partial \tilde{\phi}}{\partial x'_2}\right) + \text{background}$$



SGS model for subgrid flux of a passive scalar

- Average over cylinder, $R_1 = \Delta$, stiring time T and pdf of Γ

$$g_1' + i\, g_2' = \frac{1}{\pi R_1^2 T} \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{R_1} \int_0^{T} \phi(r,\theta,t)\, i\, u_\theta\, \mathrm{e}^{i\theta}\, p(\Gamma)\, d\theta\, r dr\, dt\, d\Gamma$$

- In laboratory co-ordinates

$$g_j = -\frac{1}{2} K\, T\, (\delta_{jp} - e_j^v\, e_p^v)\, \frac{\partial \tilde{\phi}}{\partial x_p}, \qquad K = \frac{1}{R_1^2} \int_0^{R_1} r^3\, \Omega^2(r)\, dr$$

- Assume $T = \gamma \Delta x / K^{1/2}$. Argument based on scalar, velocity structure functions then gives

$$\gamma = \frac{2}{\pi\, \beta} \left(\frac{2}{3\mathcal{K}_0} \right)^{1/2}, \quad \mathcal{K}_0 = 1.67, \quad \beta = 0.67 \rightarrow \gamma = 0.74$$

SGS model for subgrid flux of a passive scalar

- Scalar flux subgrid model - tensor diffusivity

$$
\begin{aligned}
g_i &= -\frac{\gamma\,\pi}{2\,k_c} K^{\frac{1}{2}} (\delta_{ip} - e_i^v\, e_p^v) \frac{\partial \tilde{\phi}}{\partial x_p} \\
\text{SGS} \quad T_{ij} &= K\,(\delta_{ij} - e_i^v e_j^v)
\end{aligned}
$$

- Model parameter $\gamma = 1$ (present demonstration)

- Model appropriate for $Sc = \nu/D = O(1)$

- Model suggests scalar gradient is orthogonal to small scale vorticity (Ruetsch & Ferziger, DNS, 1997).

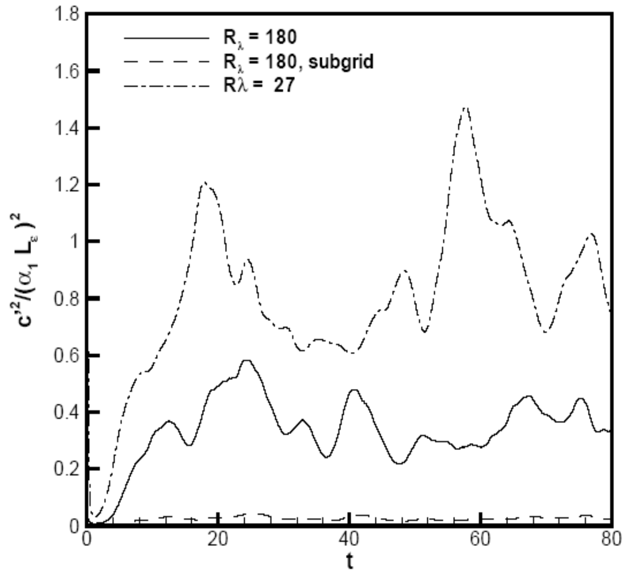
Passive scalar with imposed mean scalar gradient in forced homogeneous turbulence

- Forced turbulence in $(2\pi)^3$ box (32^3). $\tilde{\phi} = \alpha_1 x_1 + \hat{\phi}$

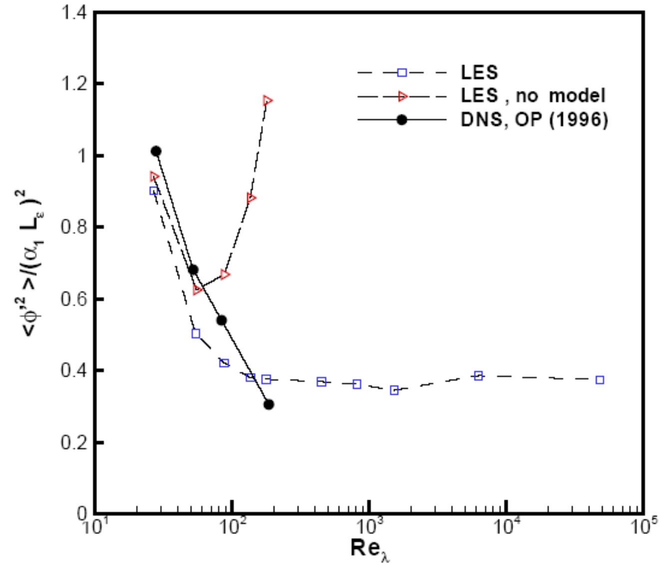
$$\frac{1}{2}\frac{\partial}{\partial t}\left\langle \hat{\phi}^2 \right\rangle + \alpha_1 \left\langle \hat{\phi}\,\tilde{U}_1 \right\rangle = \left\langle g_i \frac{\partial \hat{\phi}}{\partial x_i} \right\rangle - D\left\langle \left(\frac{\partial \hat{\phi}}{\partial x_i} \right)^2 \right\rangle$$

- α_1 is preserved by the evolution
- Statistical steady state for $\left\langle \hat{\phi}^2 \right\rangle$
- DNS, $R_\lambda = 27 - 180$ [Overholt and Pope, **Phys Fluids**, 1996]

Passive scalar with imposed mean scalar gradient in forced homogeneous turbulence



Scalar variance $< \phi'^2 > / (\alpha_1 L_\epsilon)^2$ versus t. $Sc = 0.7$
$L_\epsilon = u'^3/\epsilon$, $T_L = L/u' \approx 2.2$



Scalar variance, 32^3 LES compared to
DNS - Overholt & Pope (1996), $32^3 - 256^3$. $Sc = 0.7$.

Velocity and resolved-scalar spectra

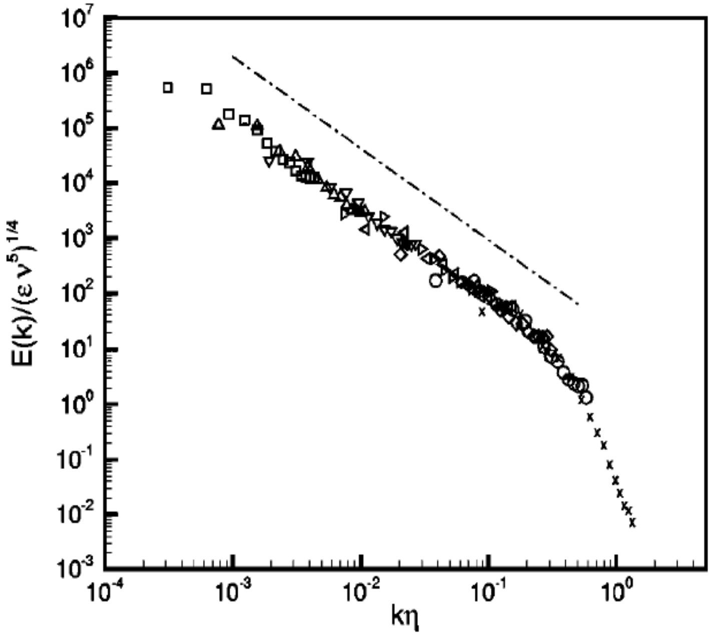


FIG. 7. Resolved flow energy spectra, LES (spiral). Cross: $Re_\lambda = 27$. Circle: $Re_\lambda = 52$. Diamond: $Re_\lambda = 84$. Left triangle: $Re_\lambda = 134$. Right triangle: $Re_\lambda = 180$. Inverted triangle: $Re_\lambda = 444$. Triangle: $Re_\lambda = 810$. Square: $Re_\lambda = 1540$. Dash-dotted line: slope $-5/3$.

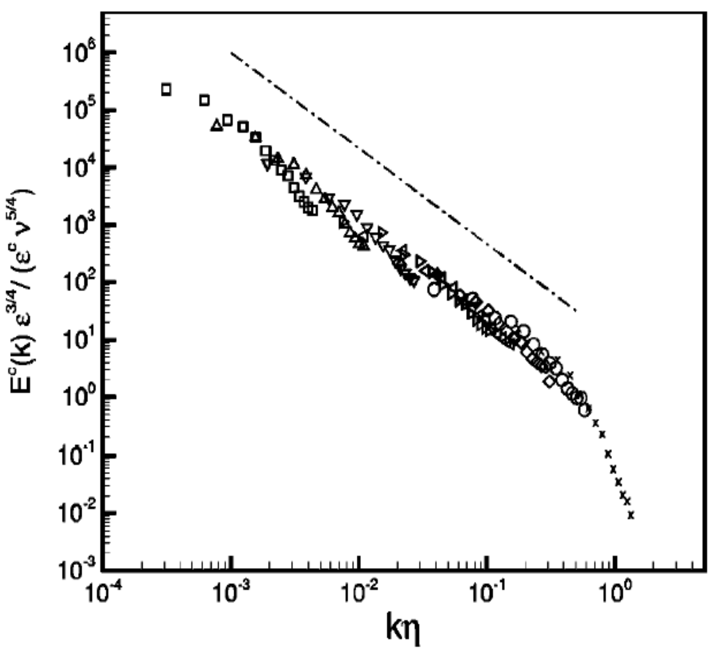


FIG. 8. Resolved flow scalar spectra, LES (spiral). For key see Fig. 7.

Compressible flow: Favre-filtered Navier-Stokes equations

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \overline{\rho}\tilde{u}_j}{\partial x_j} = 0$$

$$\frac{\partial \overline{\rho}\tilde{u}_i}{\partial t} + \frac{\partial(\overline{\rho}\tilde{u}_i\tilde{u}_j + \overline{p}\delta_{ij})}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j}$$

$$\frac{\partial \overline{E}}{\partial t} + \frac{\partial(\overline{E} + \overline{p})\tilde{u}_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\overline{\kappa}\frac{\partial \overline{T}}{\partial x_j}\right) + \frac{\partial \sigma_{ji}\tilde{u}_i}{\partial x_j} - \frac{\partial q_j^T}{\partial x_j}$$

$$\frac{\partial \overline{\rho}\tilde{\psi}}{\partial t} + \frac{\partial(\overline{\rho}\tilde{\psi}\tilde{u}_j)}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\overline{\rho}\tilde{D}\frac{\partial \tilde{\psi}}{\partial x_j}\right) - \frac{\partial q_j^\psi}{\partial x_j}$$

$$\begin{aligned} \tau_{ij} &= \overline{\rho}(\widetilde{u_i u_j} - \tilde{u}_i\tilde{u}_j) \\ q_j^T &= \overline{\rho}(\widetilde{c_p T u_j} - \tilde{c_p}\tilde{T}\tilde{u_j}) \\ q_j^\psi &= \overline{\rho}(\widetilde{\psi u_j} - \tilde{\psi}\tilde{u_j}) \end{aligned}$$

$$\overline{E} = \frac{\overline{p}}{(\tilde{\gamma} - 1)} + \frac{1}{2}\overline{\rho}(\tilde{u}_k\tilde{u}_k) + \frac{1}{2}\tau_{kk}$$

$$\overline{p} = \frac{\overline{\rho}R\tilde{T}}{\tilde{m}}$$

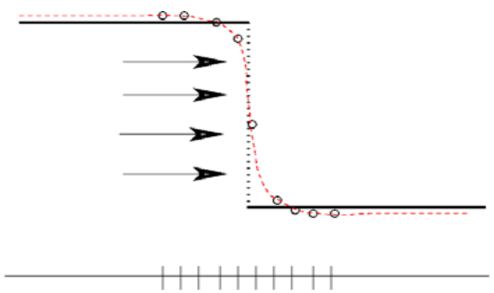
- Two-component Favre-filtered NS equations
- Filtering procedure strictly formal
  - *Not performed explicitly in LES*
  - *Guide to SGS modeling*
- Favre-filtered quantities identified with resolved-scale quantities in LES
  - *Modeling assumption on par with SGS modeling*
- Model SGS temperature flux as passive scalar

LES of compressible turbulence. LES and strong shocks (D. Hill). Hybrid WENO-TCDS algorithm:

- Numerical methods for shock-capturing and LES `orthogonal'.
- Our solution: hybrid technique: blending Weighted Essentially Non-Oscillatory (WENO) scheme with Tuned Centered-Difference (TCD) stencil.
- WENO in regions of very-large density ratio (Shocks)
  - *But WENO is not suitable for LES in smooth regions away from shocks.*
  - *Upwinding strategy is too dissipative*
- Tuned center-difference (TCD) stencil in smooth regions away from shocks
  - *Low numerical dissipation (centered method)*
  - *optimized for minimum resolved-scale discretization error in LES (Ghosal, 1996)*
  - *5- or 7-point stencil trades off formal order of accuracy for small dispersion errors*
- Target WENO stencil = TCD stencil
- In practice, target TCD stencil not always achieved; switch is used based on acceptable WENO smoothness measure
- Hybrid method designed for LES in presence of strong shocks

Shock capturing solvers; WENO

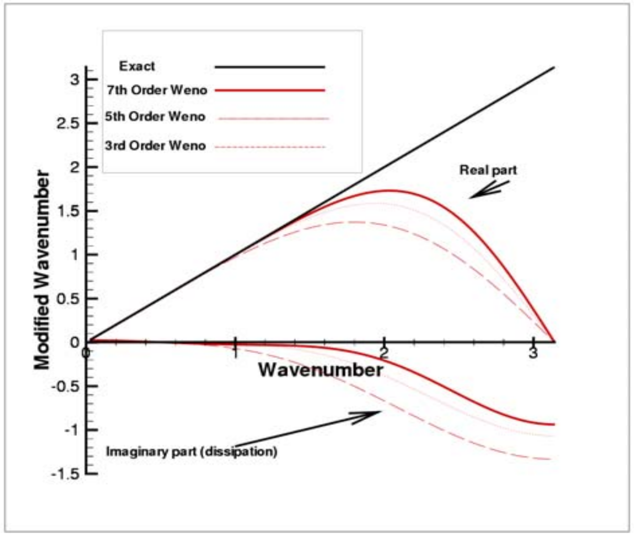
- True shocks have a thickness on the mean free path order
- The shocks are not resolved: Euler equations are solved in conservative form
- Euler solver shocks are ‘captured’, I.e. smeared across a few cells – first-order accurate at shocks



$$\frac{d\mathbf{q}}{dt} + \frac{\partial \mathbf{F(q)}}{\partial x} + \frac{\partial \mathbf{G(q)}}{\partial y} + \frac{\partial \mathbf{H(q)}}{\partial z} = 0$$

$$\mathbf{q} = \left(\rho, \rho u, \rho v, \rho w, E\right)^T$$

$$\mathbf{F(q)} = \begin{pmatrix} \rho u \\ \rho u^2 + P \\ \rho uv \\ \rho uw \\ \rho u(E+P) \end{pmatrix}$$



Weighted Essentially Non-Oscillatory (WENO) method (Osher)

Tuned Center-Difference Stencil (TCD)

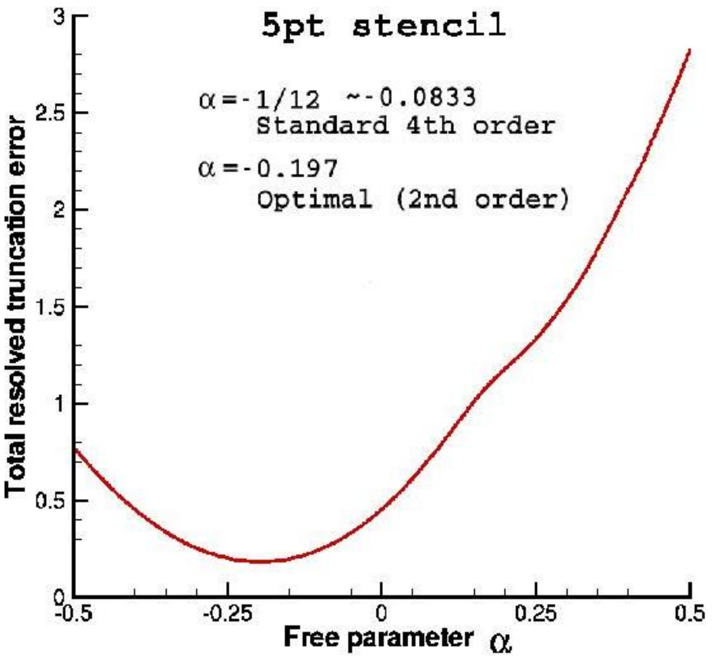
- Ghosal (JCP, 1996)
- Error in resolved-scale energy spectrum produced by one step of Navier-Stokes equations using given discretization
- Asssume –
  - *Von-Karman energy spectrum*
  - *Joint normal velocty pdf*

- $\mathcal{E}^{(\mathrm{FD})}(\kappa,\tilde{\kappa}(\kappa,\alpha))$ is spectrum of truncation error for numerical method with modified wavenumber behavior $\tilde{\kappa}(\kappa,\alpha)$

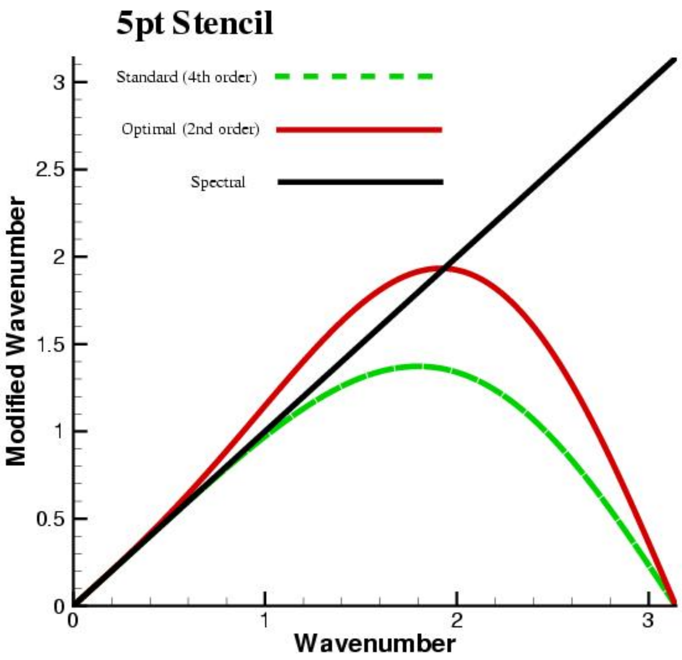
- Define total discretization error

$$\mathrm{E}_G(\alpha) = \int_0^{\frac{\pi}{\Delta x}} \mathcal{E}^{(\mathrm{FD})}(\kappa,\tilde{\kappa}(\kappa,\alpha))\,d\kappa$$

Optimized 5-point TCD stencil (second order)

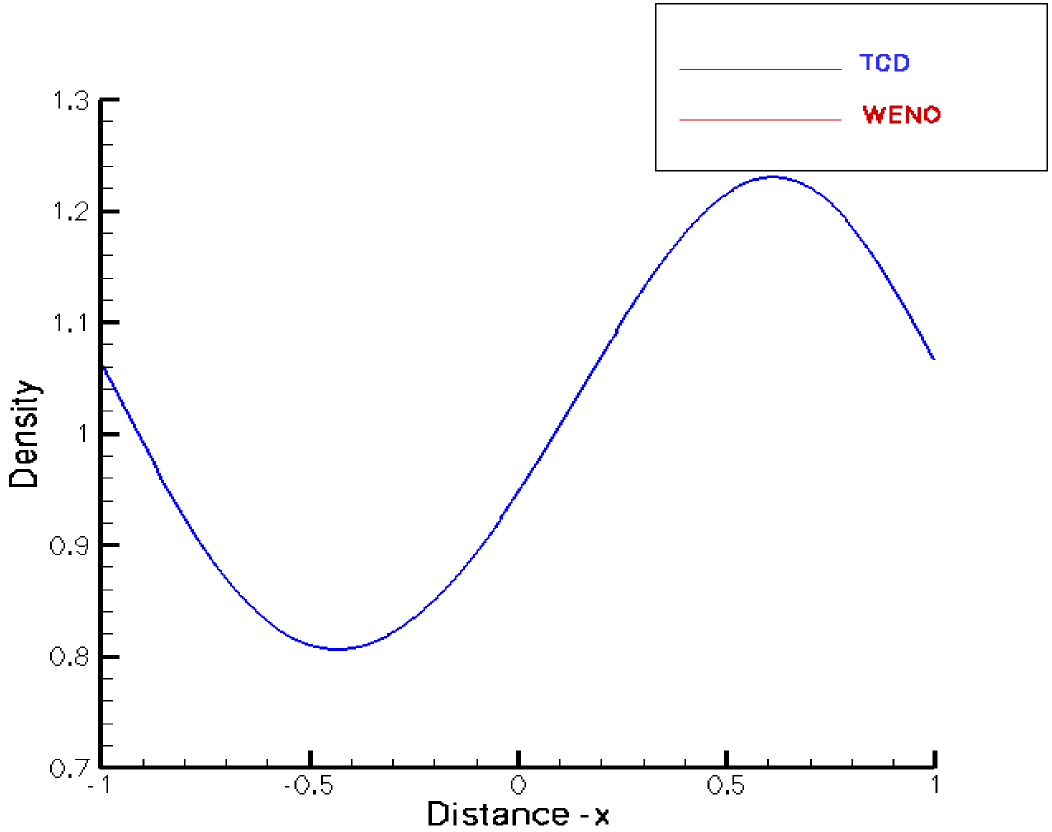


Truncation error



Modified wavenumber of minimal error stencil

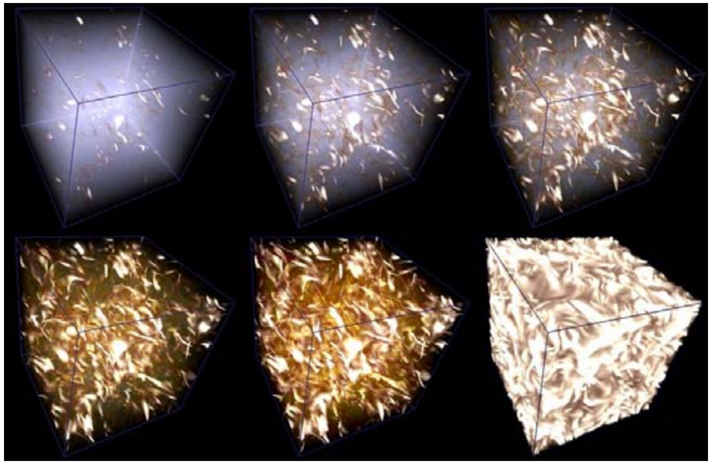
Test I: Riemann 1D Wave Exact solution of 1D Euler



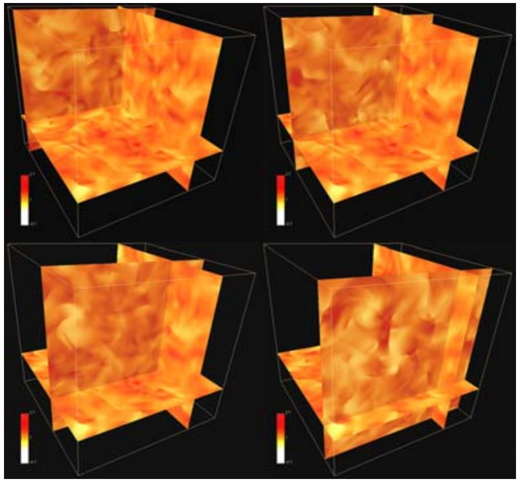
Hybrid WENO-TCD scheme

Test II; decay of compressible homogeneous turbulence

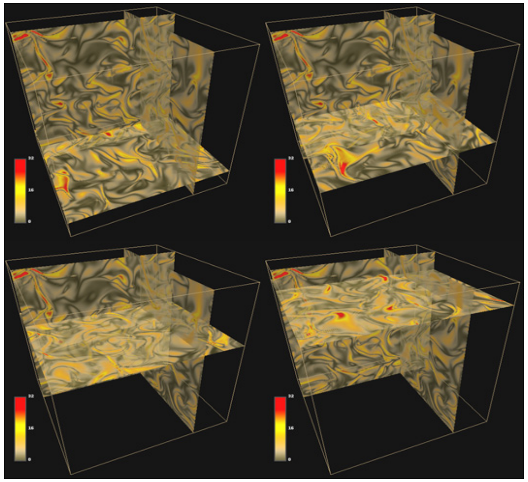
- 256^3 DNS of compressible decaying turbulence. Fully resolved (R. Samtaney)
- 10-th order compact Pade scheme
- R_lambda ~70, M_t = 0.49 (shocklets in turbulence)



isosurfaces of |vorticity| (different levels)

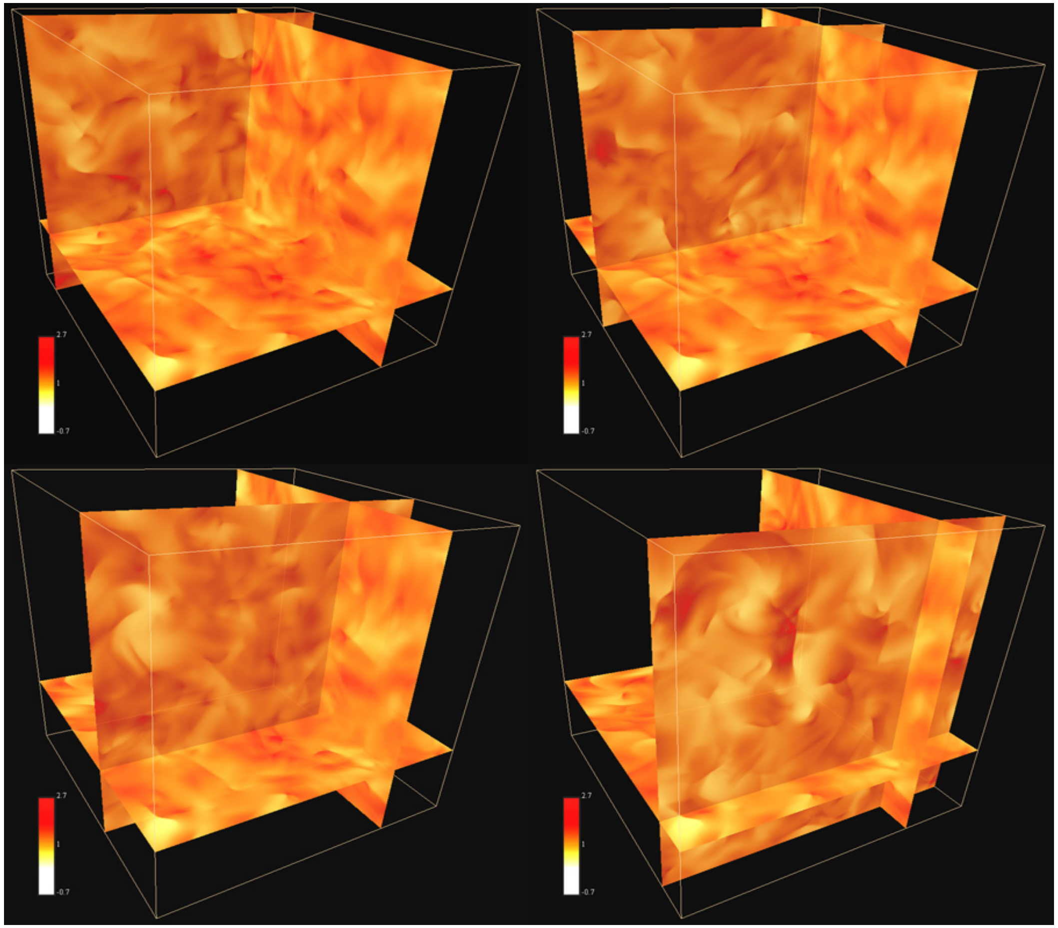


Density slices



Vorticity slices

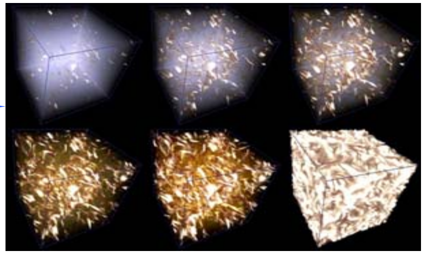
Test I; decay of compressible homogeneous turbulence

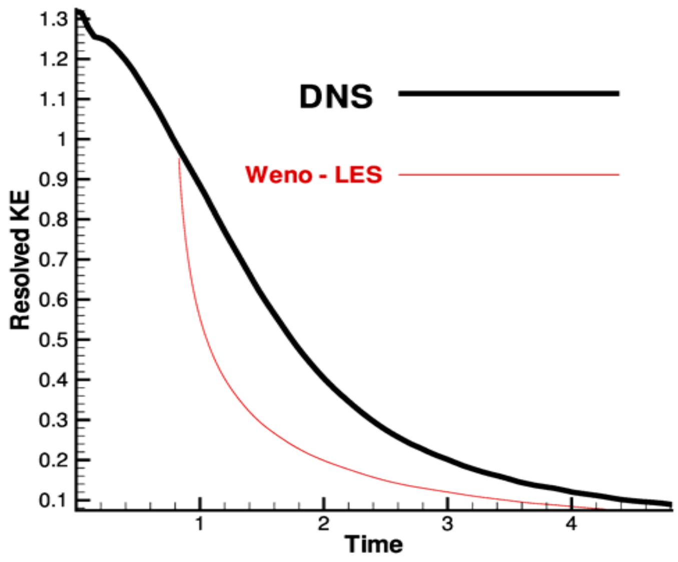


Density slices. Shocklets (weak shocks) evident

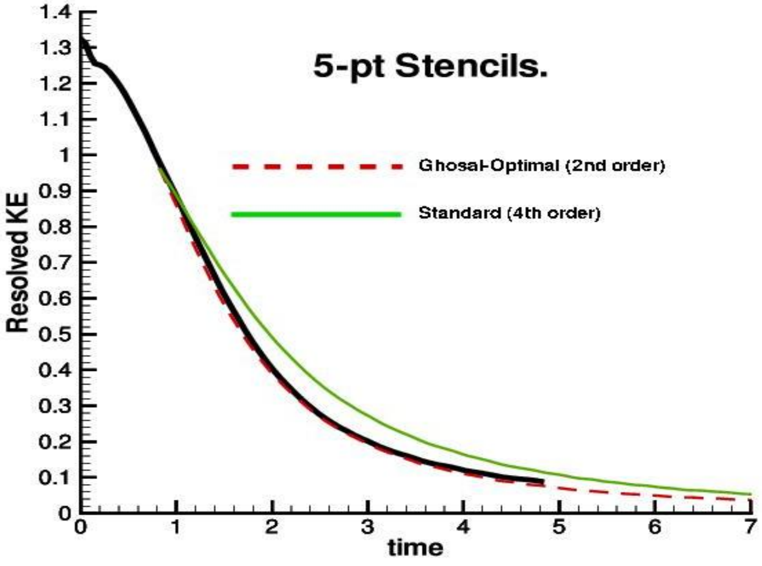
DNS and LES of decaying compressible turbulence; decay of TKE

- M_t =0.488, R_lambda = 70.
- Black; 256^3 DNS (10-th order Pade)
- Other; 32^3 LES
- Stretched-vortex SGS model





WENO scheme



Hybrid WENO-TCD scheme

Stretched-vortices for SGS modeling

- Structure-based sub-grid-scale model for LES
- Uses stretched-spiral vortex as subgrid vorticity element
- 2-point SGS statistics known (spectrum, structure function)
- Uses second order velocity structure functions to dynamically determine model parameters
- Need model for SGS vortex orientations
- Good performance for standard LES tests; decaying turbulence and channel flow at moderate Reynolds number
- Extended to compressible flow
- Problems:
  - *Each new/different SGS physics problem requires new analysis*
  - *Contains internal parameters (e.g. circulation etc.) not easily related to outer flow variables. Not needed for simple SGS momentum/scalar flux modeling but may be important for more complex SGS physics.*
- Promise; may admit SGS extension of some turbulence statistics, i.e. multi-scale modeling; effect of Schmidt number