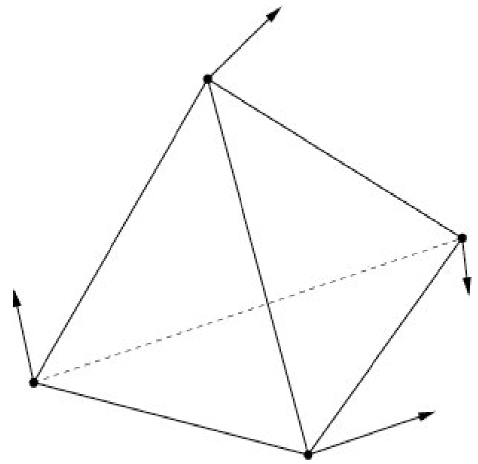
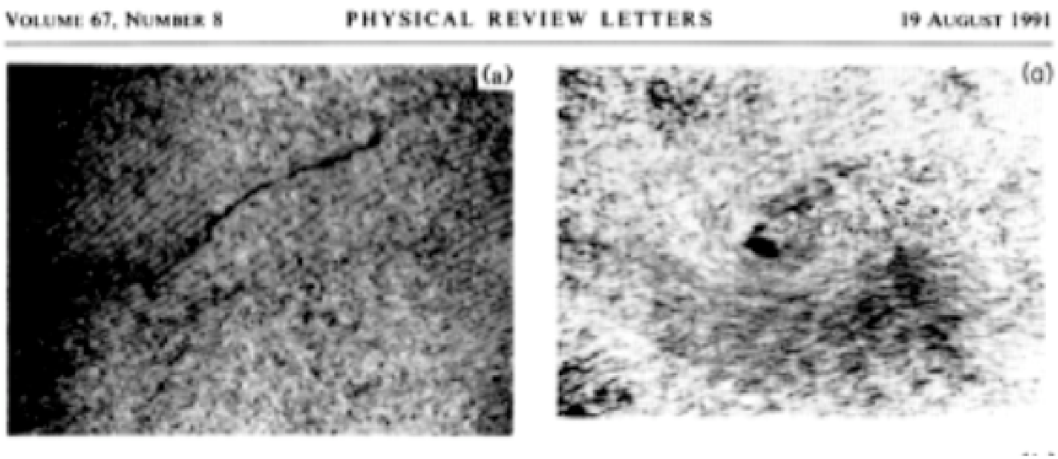
Geometry and statistics in turbulence



Alain Pumir, CNRS and Université de Nice.

Fluid turbulence occurs when fluid motion is fast (large Reynolds). It is characterized by :

- A complex ('turbulent') spatio temporal dynamics.
- The existence of a wide range of spatially excited scales (… notion of scaling).
- Despite its complexity, turbulent flows exhibit well-defined structures. Examples of vortex tubes :

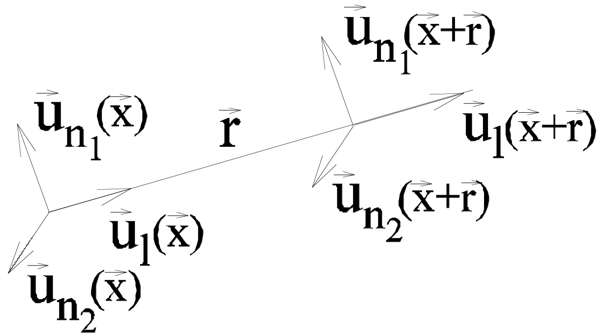


Douady et al, PRL 1991

Roughly speaking, most investigations of turbulence consider separately either of the two aspects :

The scaling properties

dependence as a function of r of the structure functions :



the local structure (i.e., the geometry) of intermittent regions in the flow (such as vorticity filaments).

⟶ Aim here : capture both aspects, by investigating multipoint correlation functions.

Objective of the work :

Develop a theoretical understanding and a description of the fluctuating velocity field that captures both the scaling and structural aspects of the flow.

An important remark :

To properly characterize the flow, focus on the full velocity gradient tensor :

$$m_{ab} = \partial_a u_b$$

Or its coarse grained generalization :

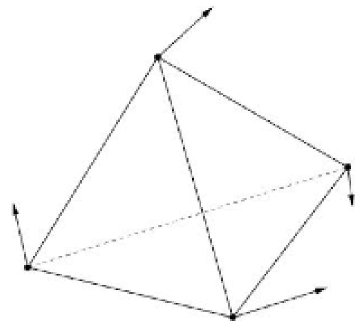
$$M_{ab} = \frac{1}{V_R} \times \int_{V_R} m_{ab} d^d r$$

The lagrangian approach has turned out to be extremely useful, in particular for the solution of the Kraichnan model (see, e.g. : Shraiman and Siggia, Nature 2000, and Falkovich et al. Rev. Mod. Phys., 2001).

It pays to just follow the flow !!

At the minimum, 4 points are needed to construct any finite difference approximation of the velocity derivative tensor, (~ M).

=> A **tethrahedron** is the minimal structure one has to study.



The evolution of the tetrahedron and M can be modelled by a stochastic differential equation (Chertkov et al, 1999) which we are studying directly.

Potential pay-offs :

- ✓ Fundamental information about the nonlinear processes in the Navier-Stokes equations.
- ✓ Invitation to think about multipoint correlation.
- ✓ Get insight about the transfer process between scales (Pumir et al, 2001, Bandi et al, 2006).
- ✓ Potentially, particle based LES (Shraiman et al, 2003).

~ new way to think about the turbulence problem

(a related approach : Chevillard + Meneveau, 2006, 2007)

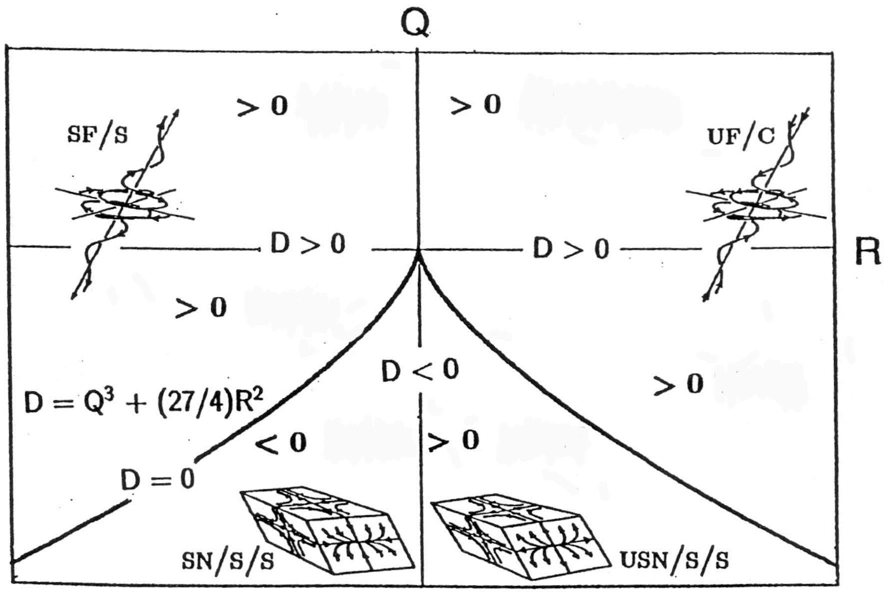
M as a diagnostic of flow topology

* The eigenvalues of M characterize the **local topology** of the flow.
* They depend (Cayley-Hamilton) on the two invariants :

$$Q = -\frac{1}{2}Tr(m^2)$$

$$R = -\frac{1}{3}Tr(m^3)$$

Vortex dominated



Strain dominated

Outline of the presentation

- The stochastic M-model : derivation and definition.
- Semi-classical solutions of the model.
- Numerical solutions and comparisons with DNS with an isotropic forcing.
- Numerical solutions in the presence of a large shear flow.
- Recent experimental results and new questions.
- Conclusions and perspectives.

The stochastic model :

Derivation and definition

The stochastic M-model : derivation and definition (1)

- Write the Navier-Stokes equation for the velocity gradient tensor :

$$\frac{dm_{ab}}{dt} + m_{ab}^2 = -\partial_{ab} p + viscosity + forcing$$

Crucial ingredient : the **pressure hessian**

- Isotropic approximation (restricted Euler dynamics, cf Vieillefosse, Cantwell) :

$$\partial_{ab} p = -\frac{1}{3} Tr(m^2)\delta_{ab}$$

The resulting system can be completely solved, with the help of the invariants Q and R ($Q = -tr(m^2)/2$; $R = -tr(m^3)/3$) :

-> **Finite time singularity** !

Singularity of the restricted Euler model

- Streamlines of the vector field :

$$dR/dt=Q^2/2$$
$$dQ/dt=-R/3$$

Divergence along the separatrix line

$$4Q^3 + 27\,R^2 = 0$$

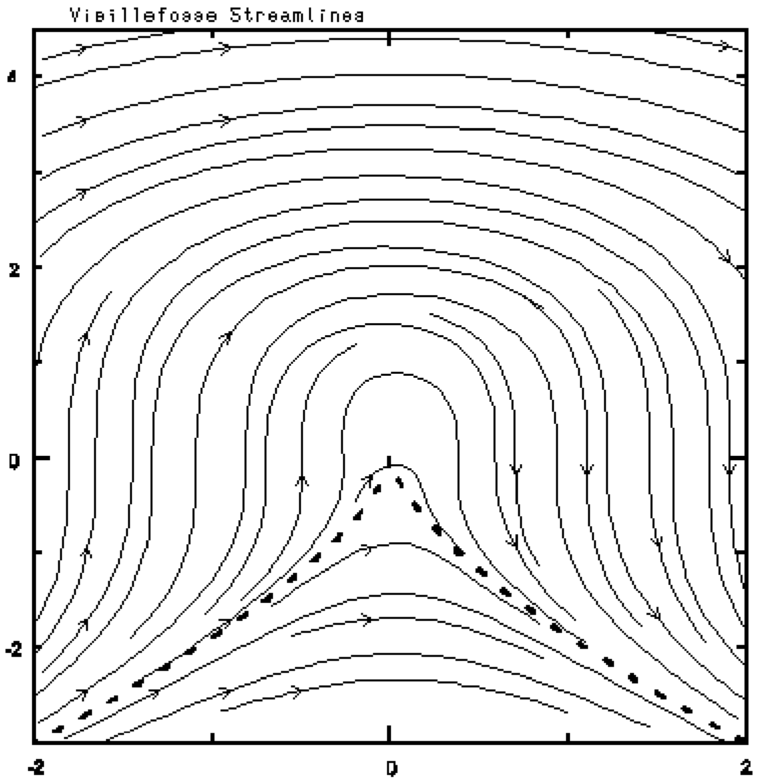


FIG. 2. The Restricted Euler flow of velocity gradient invariants $Q = -\frac{1}{2}\mathrm{tr}M^2$ and $R = -\frac{1}{3}\mathrm{tr}M^3$

The stochastic M-model : derivation and definition (2)

- To go beyond the Vieillefosse singularity, one needs to introduce the **geometry** of the Lagrangian set of points.

Equation for the geometry, derived from :

$$\frac{d\rho}{dt} = v = \rho.M + \xi$$

Where : $\rho.M$ = coherent component of the velocity field (k~1/R)

ξ = rapidly fluctuation component (k >> 1/R).

$\rho_i^a = (\vec{\rho}_i)^a$ = set of reduced coordinates, parametrizing the tetrad.

Introduce the **moment of inertia tensor** : $g = \rho^t \rho$

The stochastic M-model : derivation and definition (3)

- Equation for the **coarse-grained velocity gradient tensor** (obtained from an approximation of the pressure Hessian, based on analytical and numerical results) :

$$\frac{dM}{dt} + (M^2 - \Pi Tr(M^2)) = \alpha(M^2 - \Pi Tr(M^2)) + \eta$$

$(M^2 - \Pi Tr(M^2))$ « local » component of the pressure

$\alpha(M^2 - \Pi Tr(M^2))$ « non local » component of the pressure.

η fluctuating component

$$\left(\Pi = \frac{g^{-1}}{Tr(g^{-1})}\right)$$

- Reduction of the nonlinearity through the pressure Hessian : the importance of this effect is measured by α

The stochastic M-model : derivation and definition (4)

- One finally obtains the following system of stochastic differential equations :

$$\begin{cases} \frac{dM}{dt} + (1-\alpha)(M^2 - \Pi Tr(M^2)) = \eta \\ \frac{d\rho}{dt} - g.M - M^t.g - \beta\sqrt{Tr(MM^t)}(g - Tr(g)Id) = 0 \end{cases}$$

The effect of the noise in the g-equation is assumed to (mostly) restore the isotropy of the g-tensor. It is substituted here by the β-term.

The noise η is modelled by a Gaussian white noise term, obeying the K41-scaling (ρ^2 = Tr(g)) :

$$\langle \eta_{ab}(\rho,t).\eta_{cd}(0,0) \rangle = \gamma\left(\delta_{ac}\delta_{bd} - \frac{1}{3}\delta_{ab}\delta_{cd}\right)\frac{\varepsilon}{\rho^2}\delta(t)$$

The stochastic M-model : derivation and definition (5)

- Summary : the model thus reduces to a set of nonlinear, stochastic differential equations, with 3 dimensionless parameters :

- Reduction of nonlinearity by the parameter α.

- Strength of the isotropy restoring term (for the g tensor), β.

- Intensity of the fluctuations in the M-equation, γ.

Energy balance

- Define the energy at scale ρ by $E=Tr(VV^t)/2$ by :
  $V_i^a=\rho_i^a M_{ba}$

Equation of evolution of the energy :

$$\partial_t E(\rho) = -\frac{\partial}{\partial \rho_i^a}\left\langle V_i^a tr(VV^T)\right\rangle_\rho + \alpha\left\langle tr(VV^T M)\right\rangle_\rho + (coupling\ with\ small\ scales)$$

- Physical interpretation :
  $-\frac{\partial}{\partial \rho_i^a}\left\langle V_i^a tr(VV^T)\right\rangle_\rho$: large scale energy flux
  $\alpha\left\langle tr(VV^T M)\right\rangle_\rho$: eddy-damping term

(see Borue and Orszag,1998, Meneveau and Katz 2000,…)

The model provides a way to compute the statistical properties of the M-tensor as a function of scale !

What is the qualitative behavior of the solutions of this system of equations ?

N.b. : it depends on the three parameters : α, β and γ.

Methods of resolution

of the system

The equation satisfied by the Eulerian PDF…

- A Fokker-Planck equation for the Eulerian PDF can be derived from this stochastic system :

$$\partial_t P(M,g,t) = L.P(M,g,t)$$

- The stationary solutions must satisfy the system :

$$L.P = 0$$

$$\int dMP(M,g) = 1$$

$$P(M,g = L^2 Id) \approx \exp\left[-\frac{Tr(MM^T)}{\left(\varepsilon L^{-2}\right)^{2/3}}\right]$$ (Gaussian distribution at the integral scale)

… and its solution in terms of path integrals

- The system can be solved using Green’s functions methods :

$$P(M,g) = \int dM' \int dT\, G_{-T}(M,g \mid M',g')P(M',g')$$

(G : Green’s function; P(M’,g’) : boundary condition)

With :

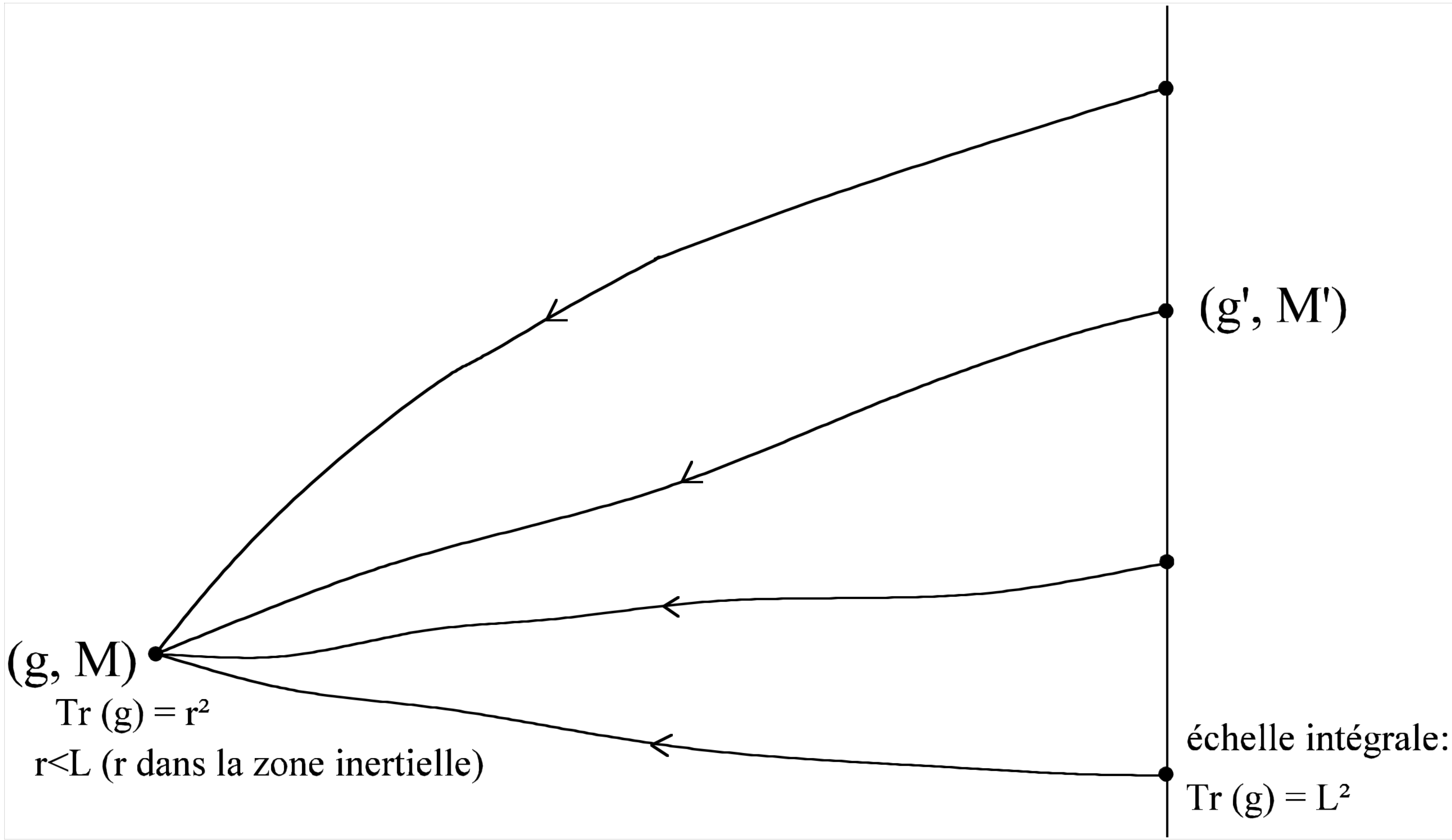
$$G_{-T}(M,g \mid M',g') = \int [DM''][Dg'']\exp\left[-S(M'';g'')\right]$$

Hence :

$$P(M,g) =$$

$$\int dM' \int dT \int [DM''] \int [Dg''] \exp\left[-S(M'';g'') + Tr(M'\,M'^{t})/(\varepsilon L^{-2})^{2/3}\right]$$

(Green’s function) (boundary condition)



Starting from an initial condition at the integral scale, one integrates the system up to a fixed scale r (in the inertial range). In principle, one has to *integrate over all trajectories* in phase space.

(Approximate) method of resolution (1)

One could use a straightforward **Monte-Carlo method** (exact in principle)

Difficulty :

the method is **extremely inefficient**, since one has to deal with trajectories with widely different statistical weight (by orders of magnitude !).

Obtaining reliable numerical results requires prohibitively large computer time.

0^{th} order approximation : look for **deterministic solutions** (γ=0)

-> encouraging results when compared with DNS (Chertkov et al, 1999)

(Approximate) method of resolution (2)

Use here the **semiclassical approximation** (saddle point approximation of the path integral)

Method : one considers only the trajectory for which the action is minimal (the one with the largest statistical weight).

Hope : The method should provide important information, especially since many trajectories do not contribute very much.

Drawback : the method is not rigorous; it is difficult to control the errors made.

=> A better algorithm has to be implemented to understand the effect of fluctuations (~Monte-Carlo), and to really estimate the errors made by using the semi-classical approximation.

Numerical solutions of the system in the semiclassical approximation with isotropic forcing.

Comparison with DNS data

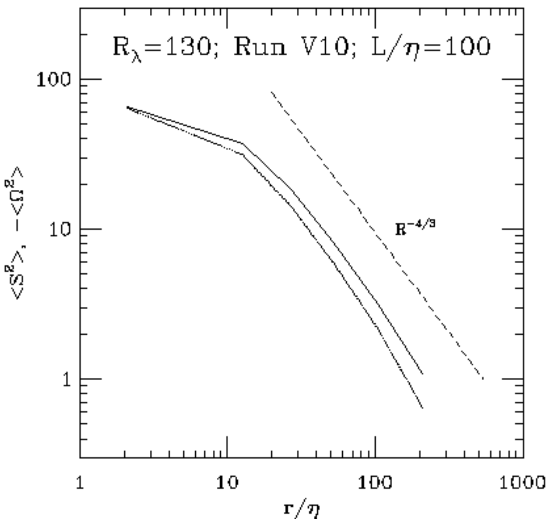
A. Naso and A. Pumir, Phys. Rev. E 72, 056318 (2005)

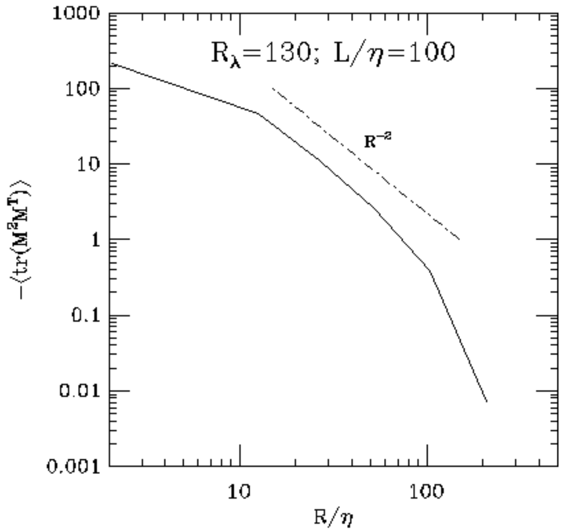
Scaling laws of the 2nd and 3rd order moments of M : DNS solutions (R_λ=130; 256^3)

According to the K41 scaling laws, $\langle \Delta u(r) \rangle \propto r^{1/3}$ so

$$\langle M(r) \rangle \propto r^{-2/3} \quad \text{and} \quad \langle \omega^2 \rangle, \langle Tr(S^2) \rangle \propto r^{-4/3} \qquad \langle -Tr(M^2 M^t) \rangle \propto r^{-2}$$

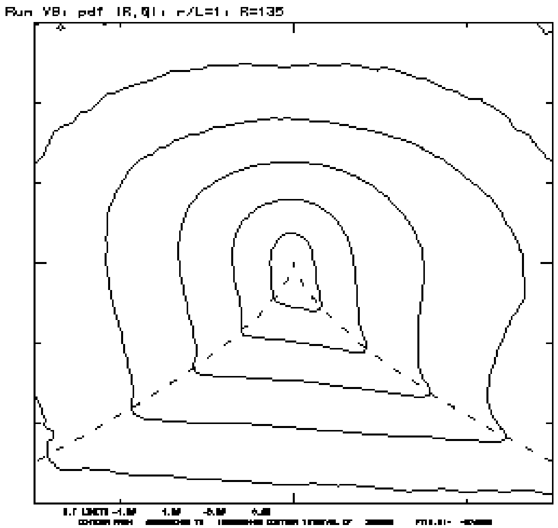
DNS results : these three quantities follow the expected Kolmogorov scaling



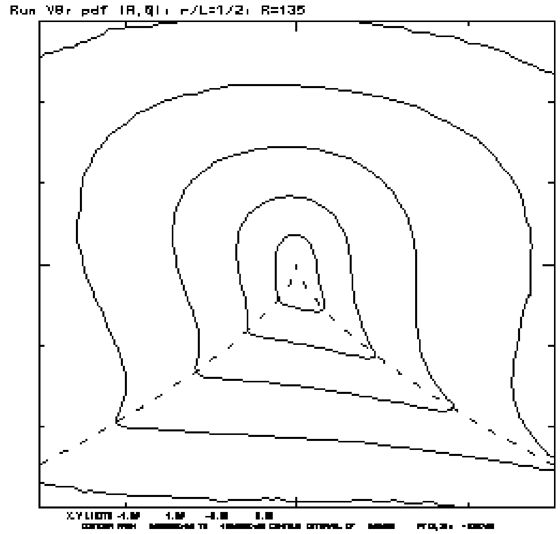


Evolution of P(R,Q) as a function of scale; DNS solutions (R_λ=130; 256^3)

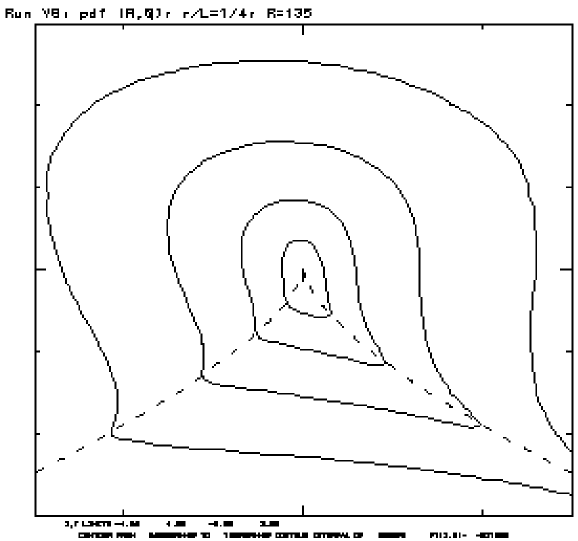
$$r/L = 1$$



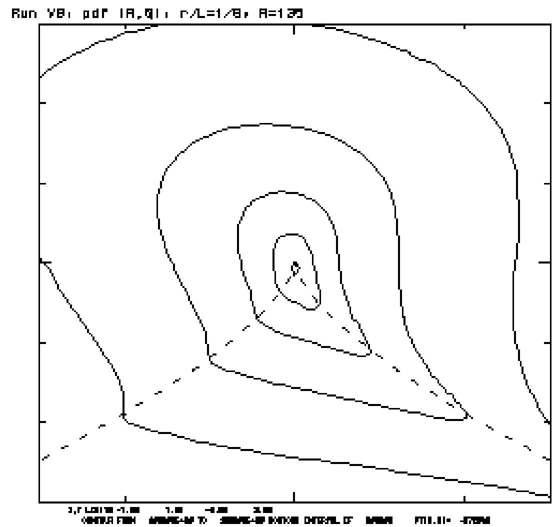
$$r/L = \frac{1}{2}$$



$$r/L = \frac{1}{4}$$



$$r/L = \frac{1}{8}$$



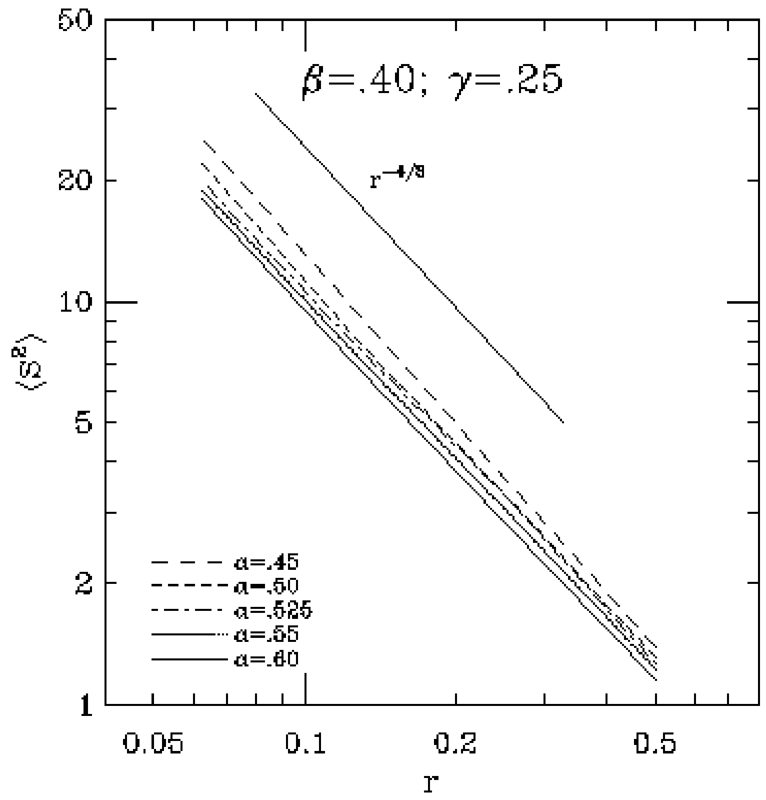
Model predictions

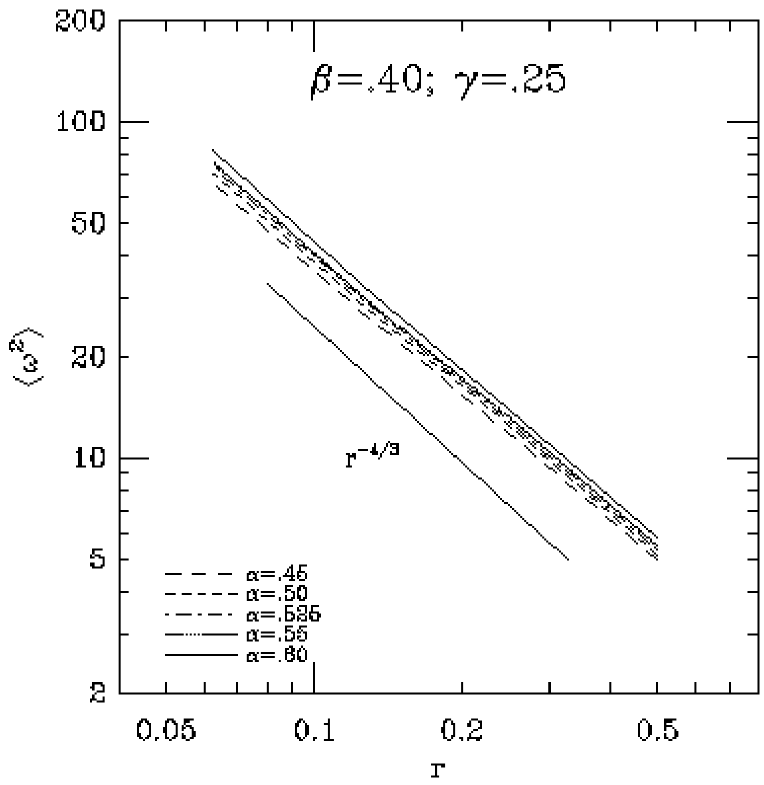
- The parameter that has the most important effect on the solution is **α**

(reduction of the nonlinearity).

- The predictions of the model agree with DNS results provided α is in a narrow interval around **$\alpha \sim 0.5$**.

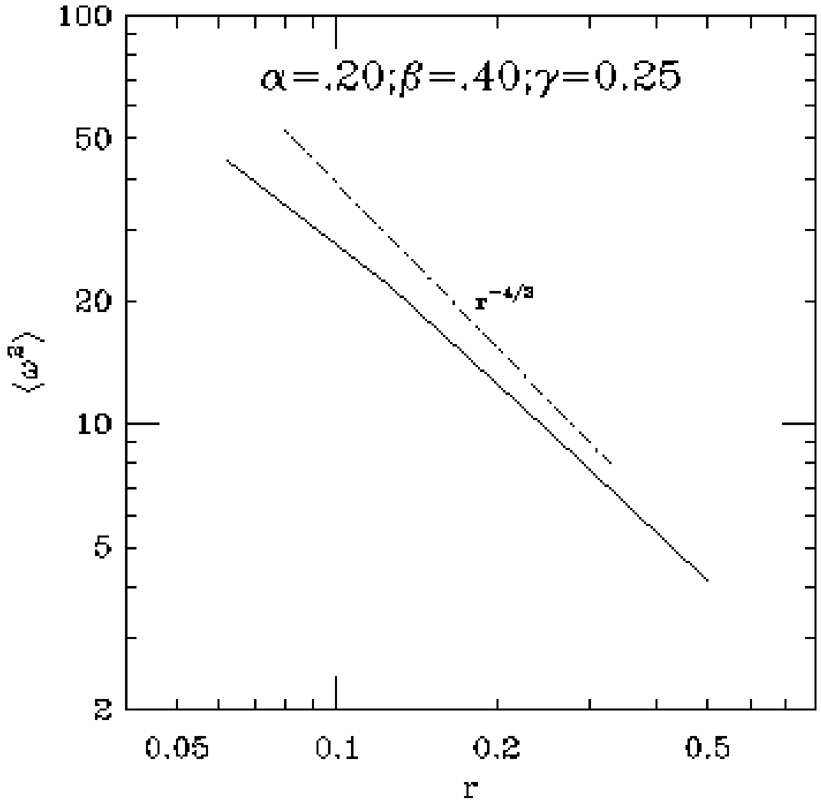
Scaling properties of the matrix M : model results (1)

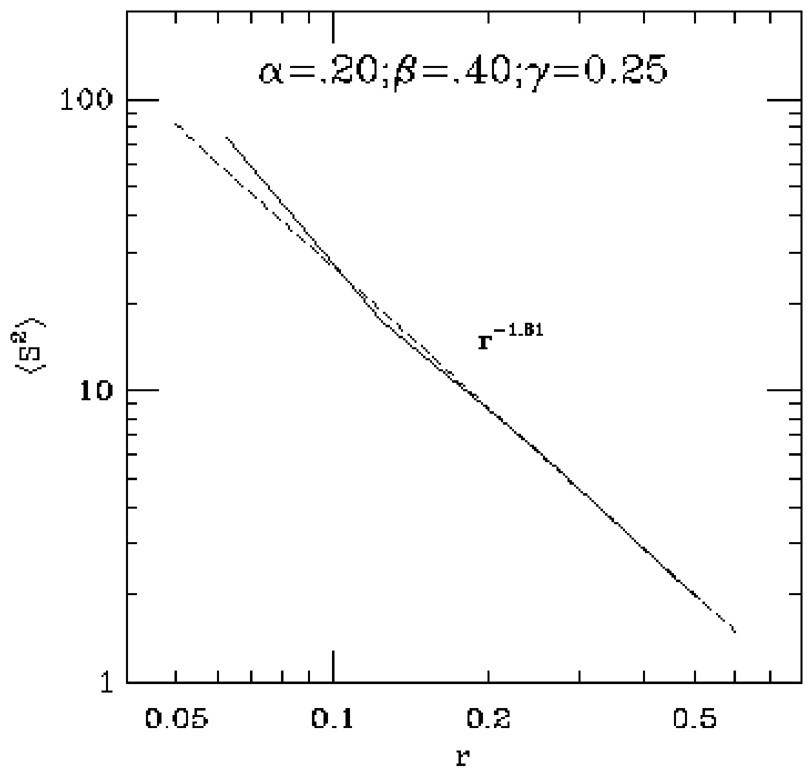




The second moment of M has the right scaling provided α is not too small !

Scaling properties of the matrix M : model results (2)

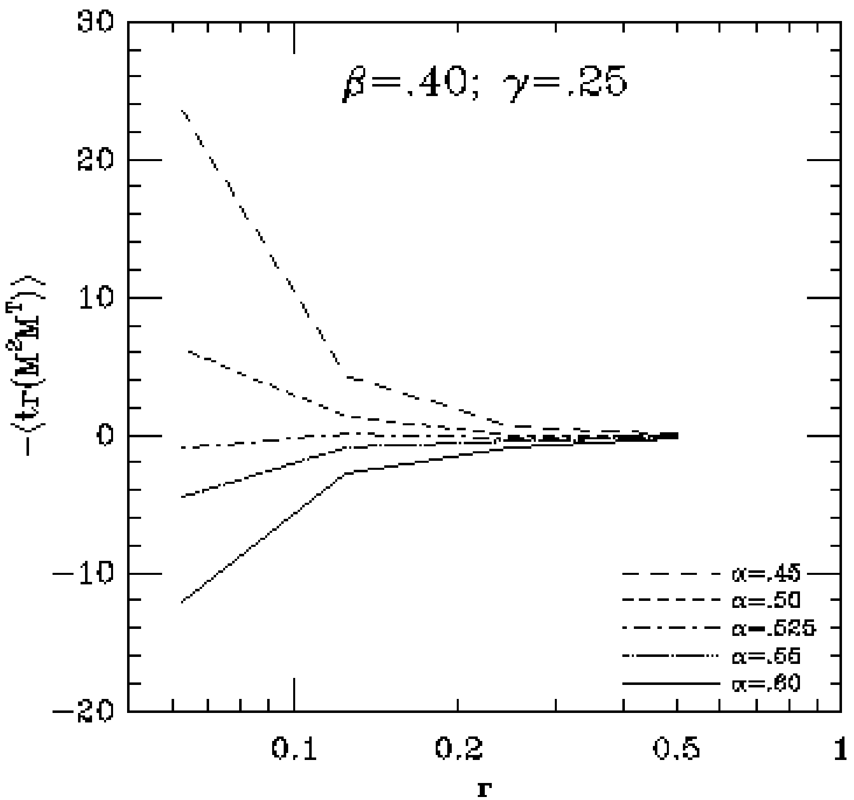




At small value of α, the strain grows with a power that differs significantly from 4/3

The **'reduction of nonlinearity'** should not be too small !

Scaling properties of the matrix M : model results (3)



The sign of <tr(M^2M^t)> is negative, as it should, for small values of α.

The **'reduction of nonlinearity'** should not be too large !

Scaling properties of the matrix M : model results (4)

Influence of the parameter β :

Not much effect provided β is large enough.

Influence of the parameter γ :

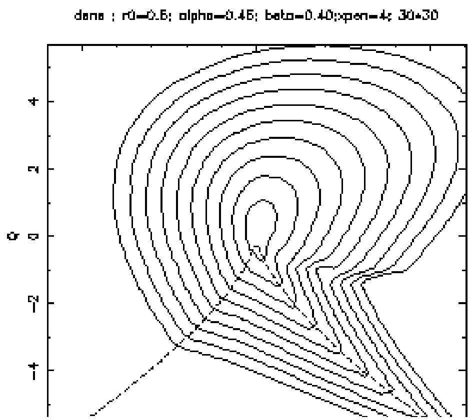
Main effect : change the numerical value of

$$\left\langle \omega^2 \right\rangle \times r^{4/3}$$

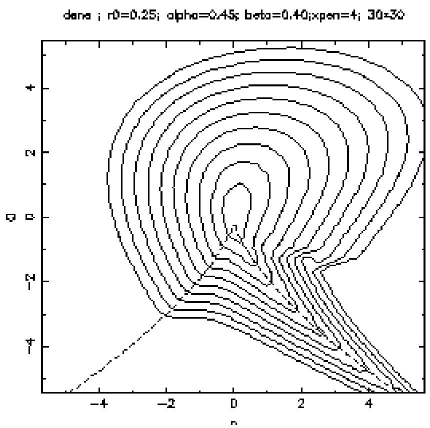
Evolution of P(R,Q) as a function of scale; semiclassical solutions of the model (1)

Parameters : $\alpha=0.45$; $\beta=0.4$; $\gamma=0.25$

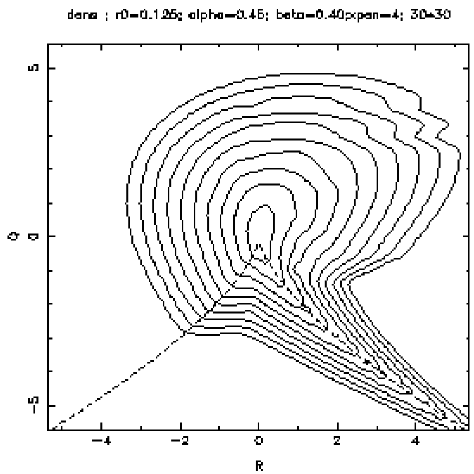
$$\frac{r}{L}=\frac{1}{2}$$



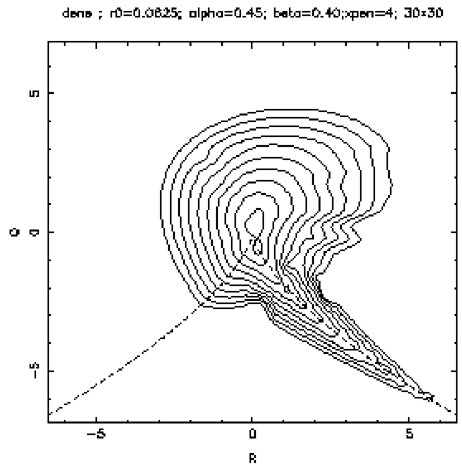
$$\frac{r}{L}=\frac{1}{4}$$



$$\frac{r}{L}=\frac{1}{8}$$



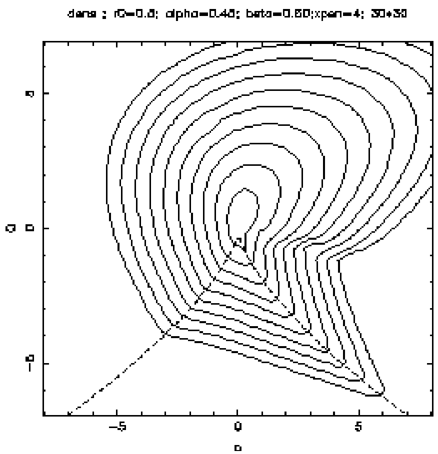
$$\frac{r}{L}=\frac{1}{16}$$



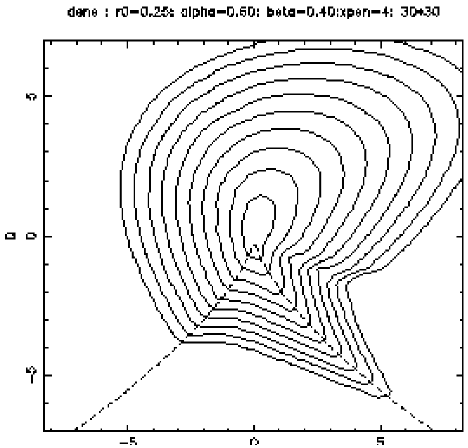
Evolution of P(R,Q) as a function of scale; semiclassical solutions of the model (2)

Parameters : $\alpha=0.6$; $\beta=0.4$; $\gamma=0.25$

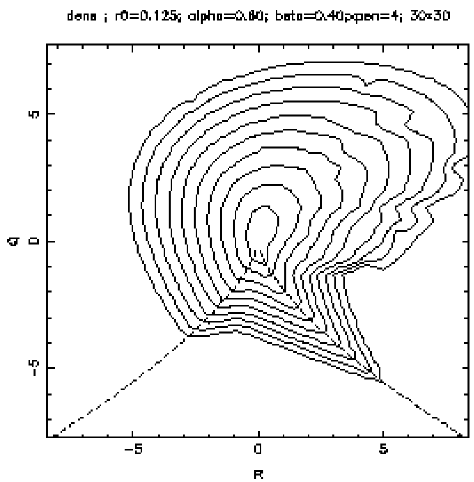
$$\frac{r}{L}=\frac{1}{2}$$



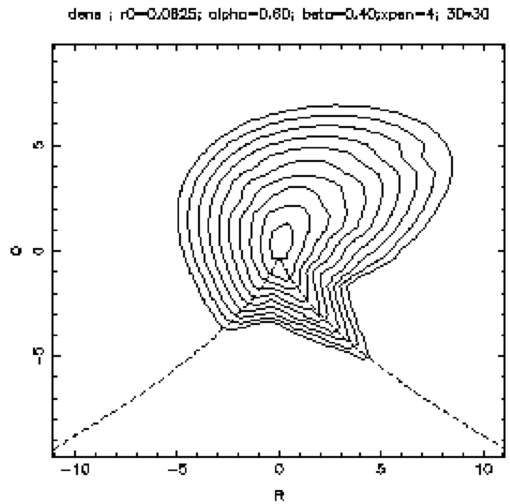
$$\frac{r}{L}=\frac{1}{4}$$



$$\frac{r}{L}=\frac{1}{8}$$



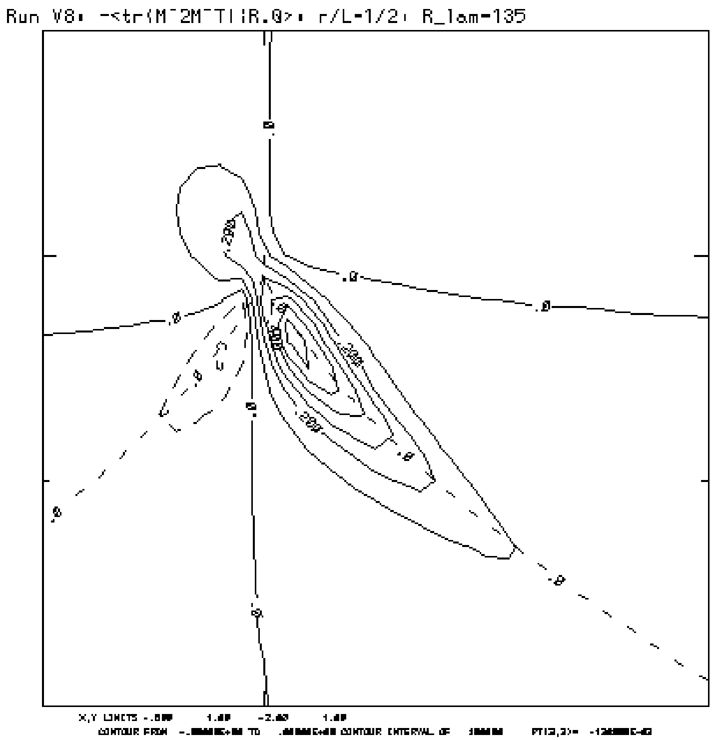
$$\frac{r}{L}=\frac{1}{16}$$

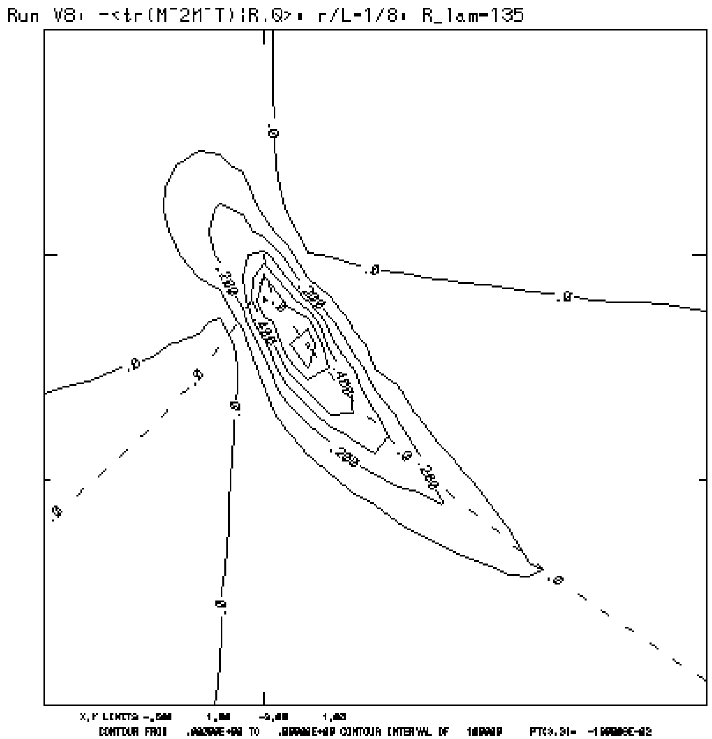


Scale dependence of the energy transfer density : DNS

$$\frac{r}{L} = \frac{1}{2}$$

$$\frac{r}{L} = \frac{1}{8}$$

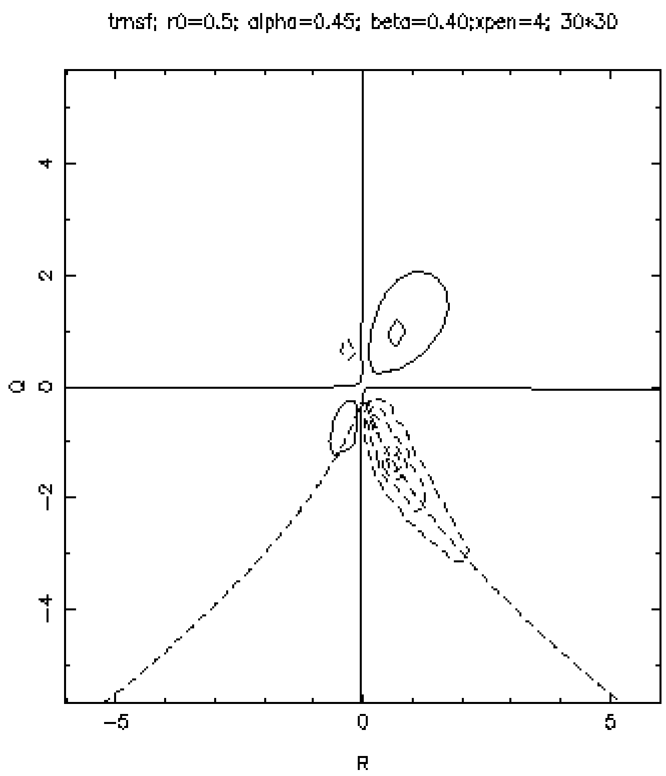


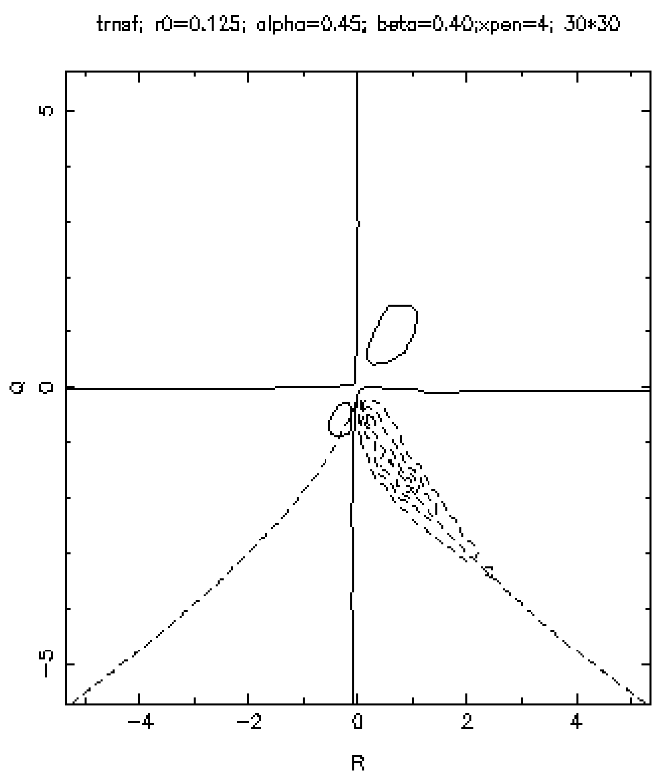


Scale dependence of the energy transfer density : semiclassical solution

$$\frac{r}{L} = \frac{1}{2}$$

$$\frac{r}{L} = \frac{1}{8}$$





Summary : acceptable values of α

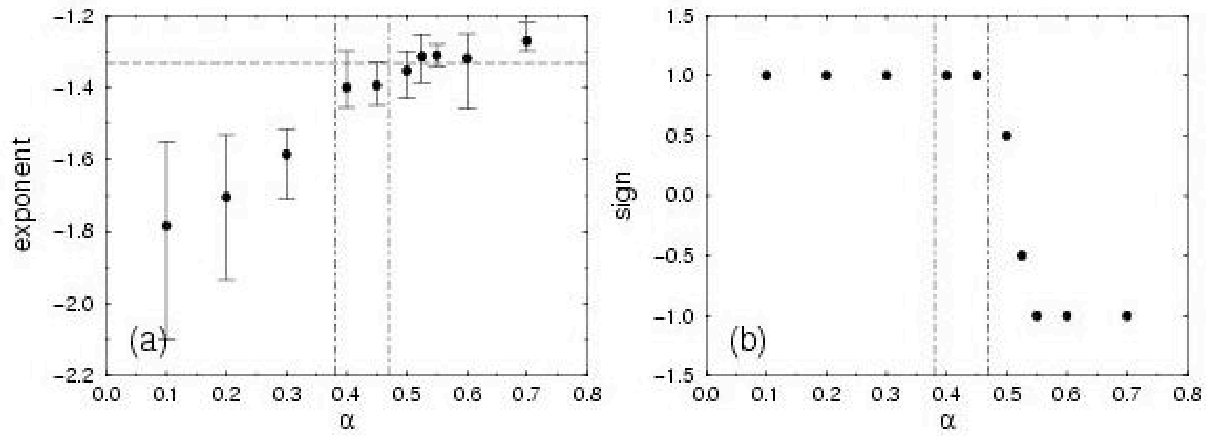


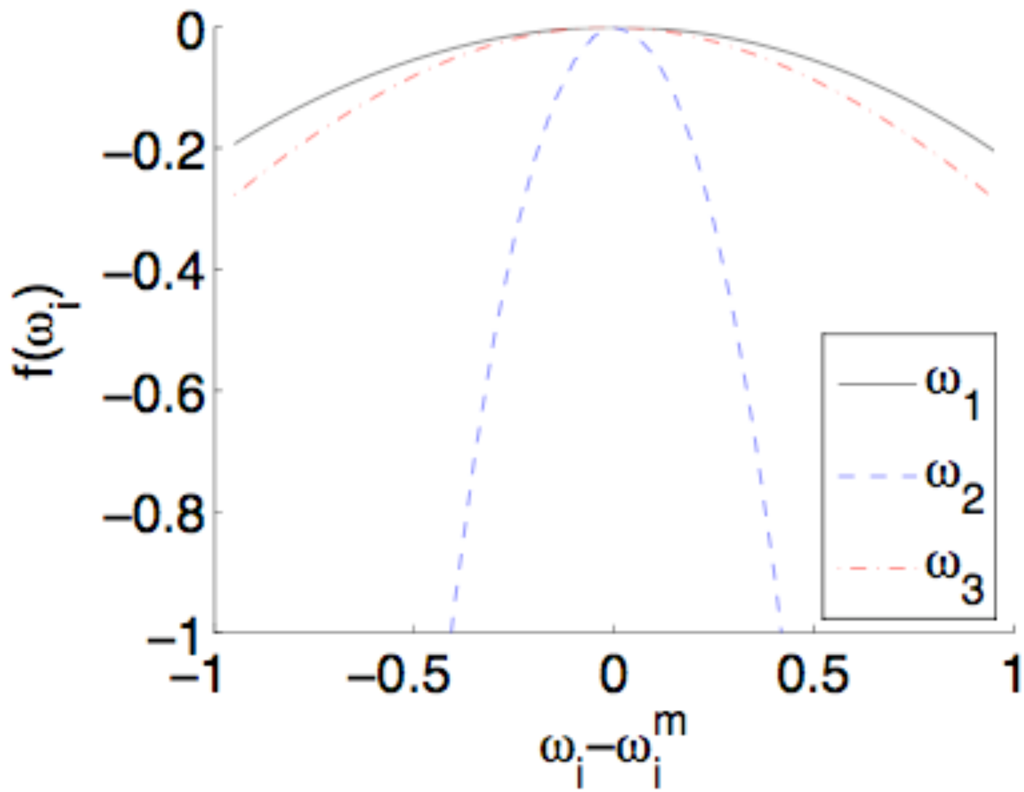
FIG. 5: Dependance with respect to α of (a) the scaling law exponent of $\langle S^2 \rangle$ and (b) the sign of the energy transfer $\langle -r^2 Tr(M^2 M^t) \rangle$ (β=0.4, γ=0.25). In (a) the dashed line indicates the Kolmogorov prediction $-4/3$. In (b) is plotted for each value of α the average of the sign of the energy transfer at the different scales considered. The range of values of α leading to a qualitatively acceptable behavior of the model's solutions is delimited by the vertical dot-dashed lines.

The solution is acceptable provided a is in a narrow interval around **α ~ 0.4-0.5 !**

Semiclassical solution : a caveat

- What we have done : determine the optimal solution, and ignore the contributions of other nearby trajectories.

- Effect of varying vorticity around the optimum : a better calculation (~MC) is necessary (A. Naso et al, 2007).



Numerical solutions of the system in the semiclassical approximation with a large scale shear.

A. Naso, M. Chertkov and A. Pumir, J. Turb. (2006)

The issue of return to isotropy

- One of the postulates of turbulence theory is the **universality of small scale** velocity fluctuations, which implies that as the scale r diminishes, the flow properties should restore **isotropy**.

- Study here an **homogeneous shear flow**.

- Nb : Experimental data (Shen and Warhaft, 2000) and numerical data (Pumir&Shraiman, 1995,1996) suggest that the return to isotropy is much slower than naively expected.

The problem studied here

- The tetrad model can be used to study several kinds of forcing, simply by changing the large scale condition.

-> impose a **large scale shear**, and calculate the scale dependence of P(R,Q), and other quantities.

- Same equations as in the isotropic case; simply change the large scale boundary condition :

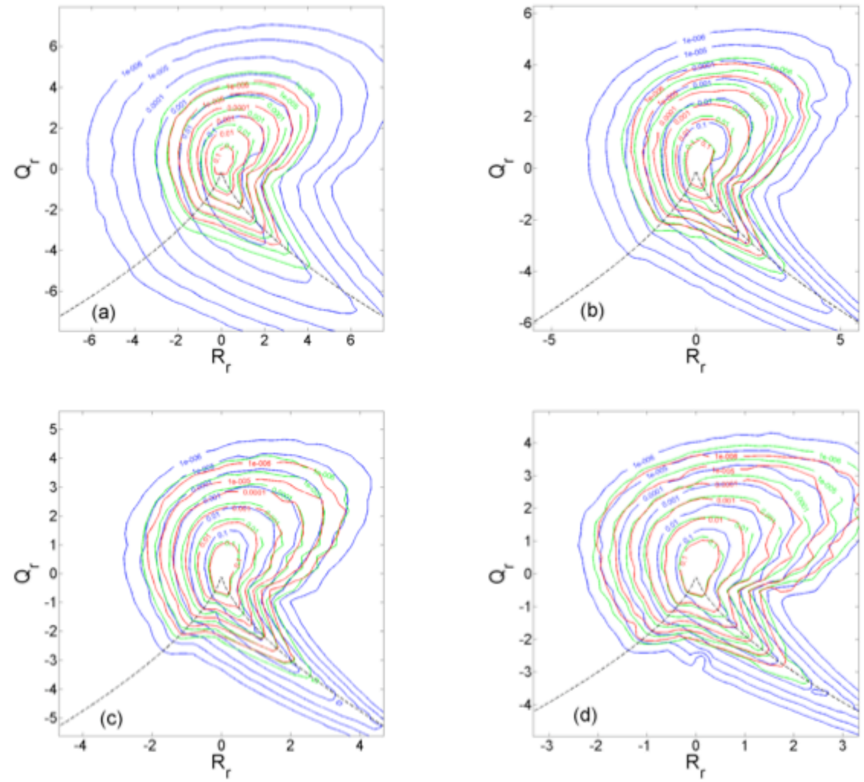
$$P(M.g = L^2 Id) \sim \exp\left[\frac{-Tr\left[(M-\Sigma)(M-\Sigma)^t\right]}{(\varepsilon L^{-2})^{2/3}}\right]$$

Where : $\Sigma = \begin{bmatrix} 0 & s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; s measures the **shear intensity**

Scale dependence of P(R,Q) : semiclassical solutions with s=0,1,6

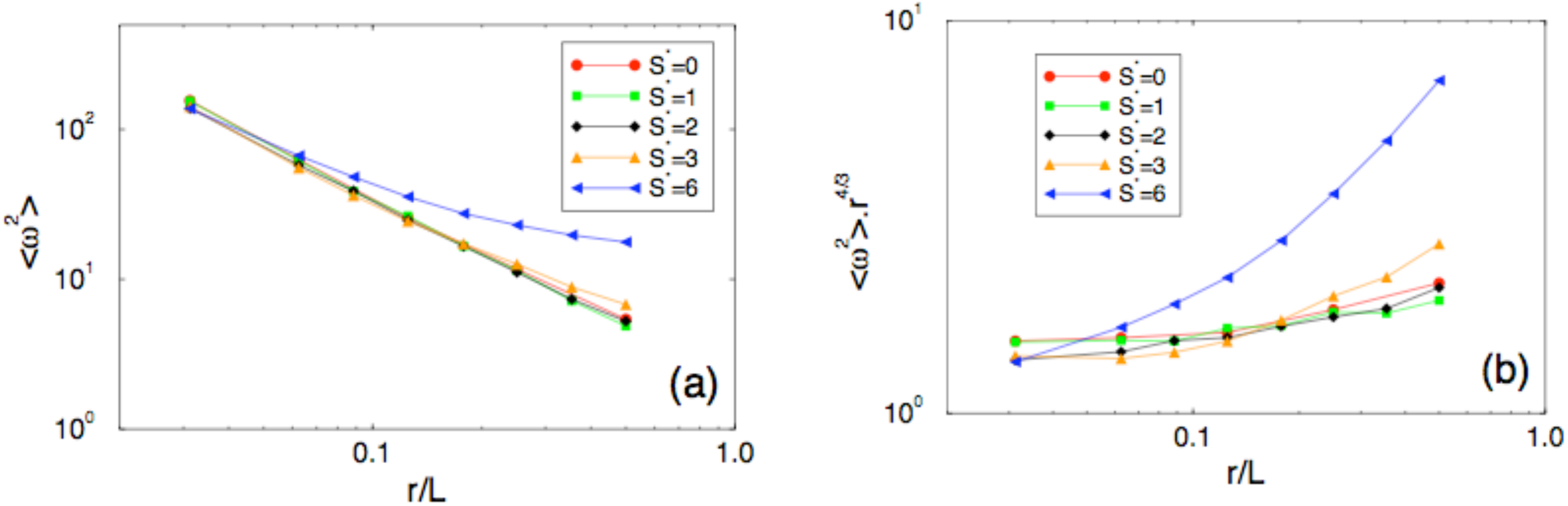
$\frac{r}{L} = \frac{1}{2}$ $\frac{r}{L} = \frac{1}{8}$

$\frac{r}{L} = \frac{1}{4}$ $\frac{r}{L} = \frac{1}{16}$

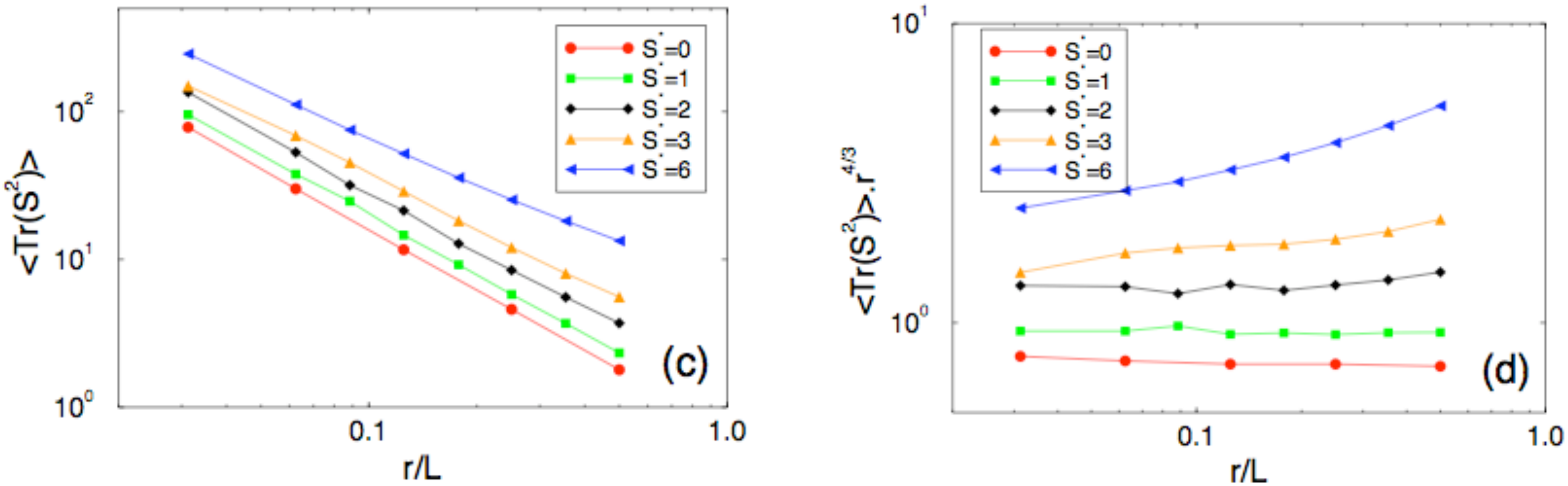


Parameters : α=0.6, β=0.4; γ=0.25

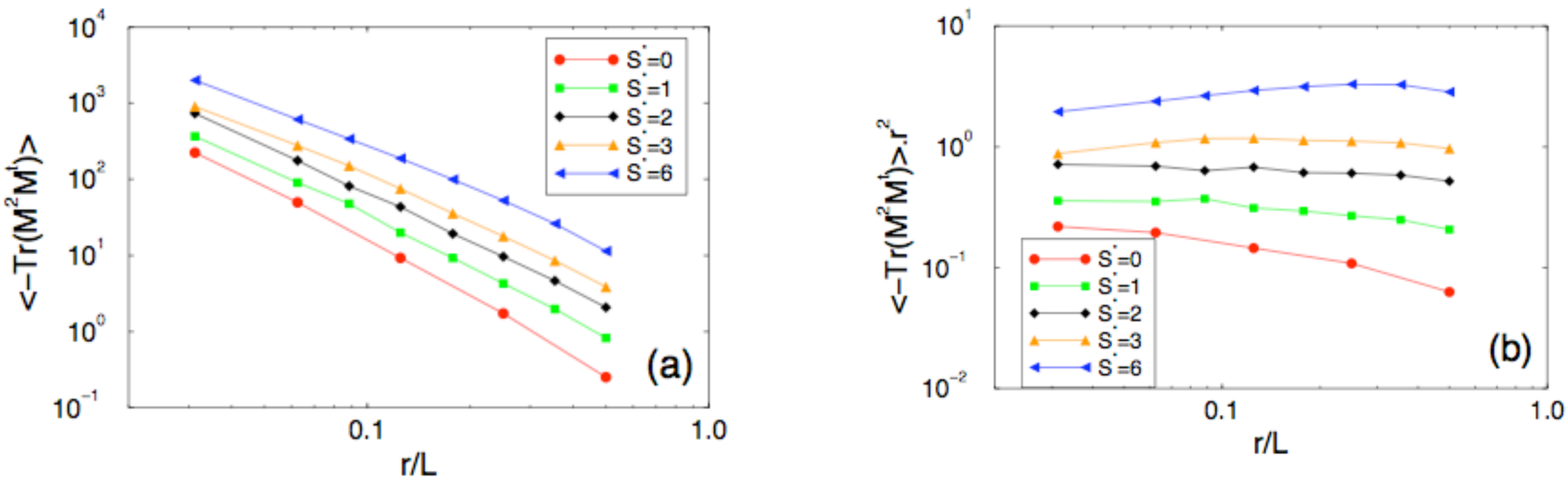
Scale dependence of $\langle\omega^2\rangle$ at different values of s



Scale dependence of <Tr(S)^2> at different values of s



Scale dependence of the energy transfer at different values of s



The issue of return to isotropy

- Our results are consistent with the accepted view that the effects of large scale anisotropy decrease when the scale decreases.

- **New finding :** difference of behavior between vorticity dominated and strain dominated structures. The anisotropy effects decrease faster for vorticity dominated quantities (enstrophy) rather than for strain dominated objects (strain, energy transfer).

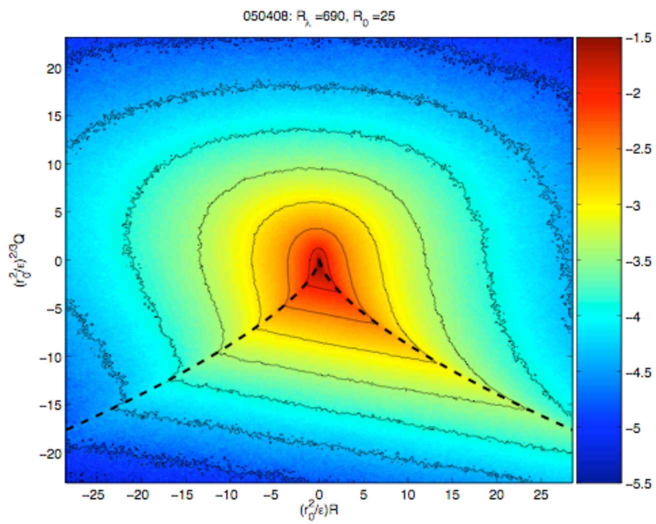
- Faster relaxation of vorticity dominated quantities towards isotropy may be consistent with the facts that vorticity is found to be more intense, hence less sensitive to the large scale forcing.

New experimental developments and new questions.

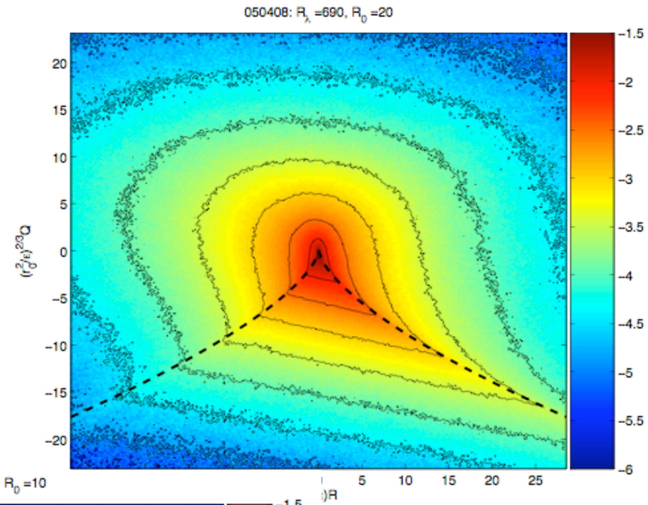
Experimental results : pdf(R,Q)

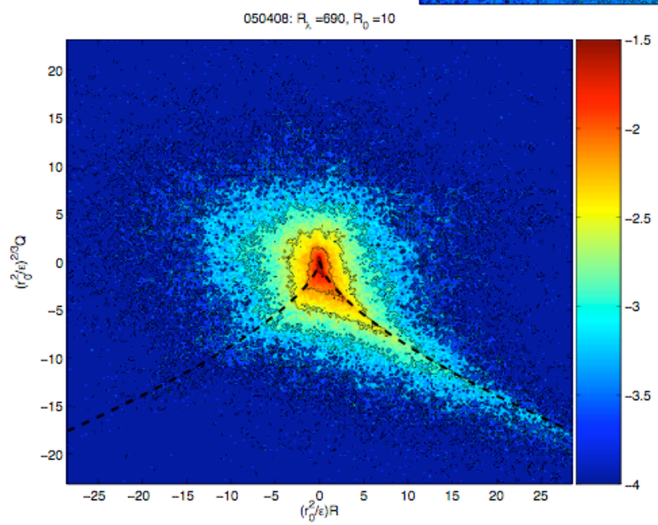
L=70mm; η=0.03mm

r_0=25mm



r_0=20mm





r_0=10mm

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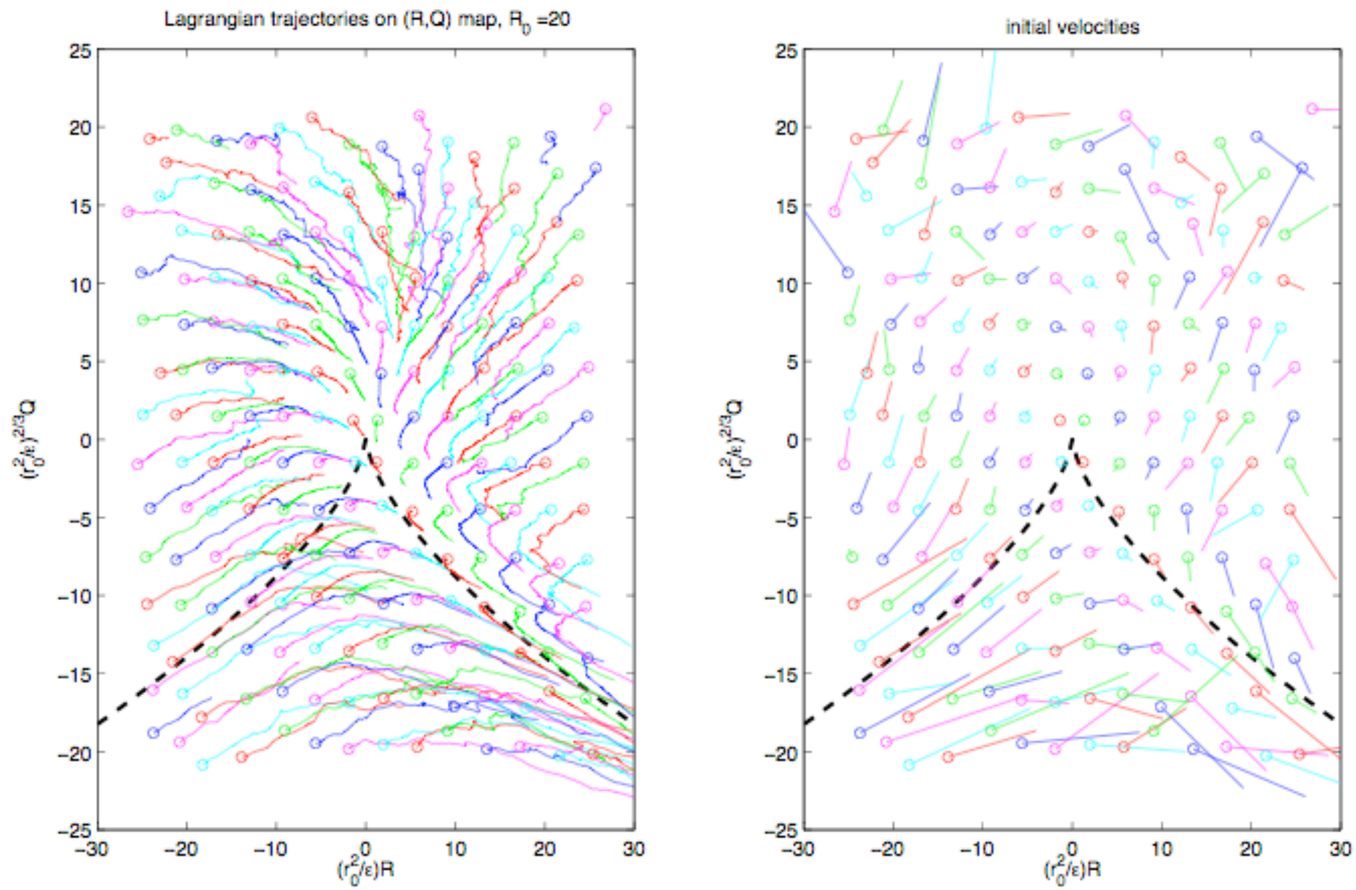
Preliminary results

Experimental results : Trajectories in the (R,Q) plane



R_λ = 690; r_0=20mm

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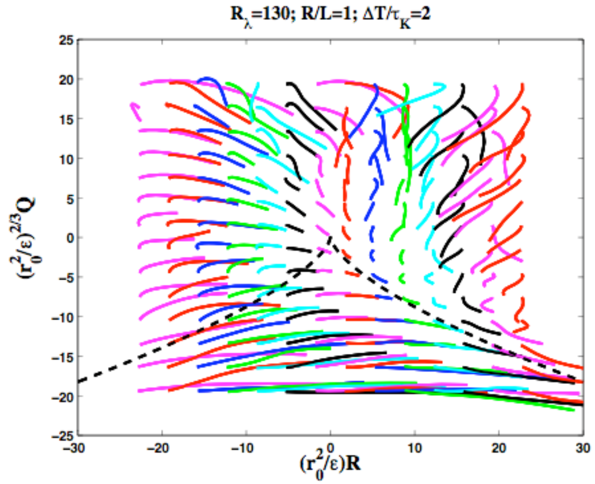
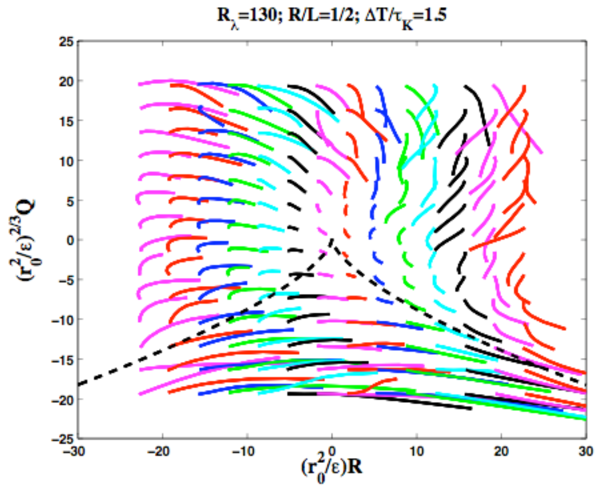
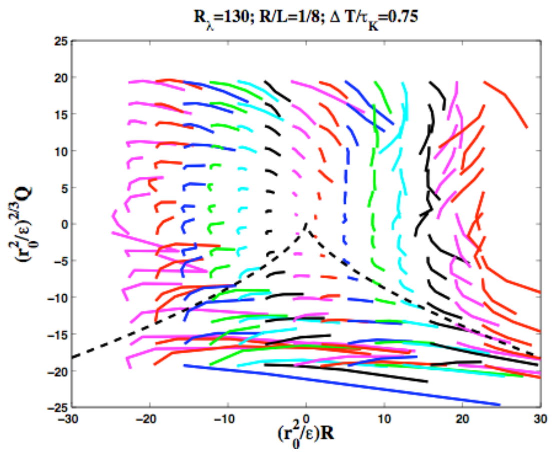
Numerical results : Trajectories in the (R,Q) plane



$r_0/L=1$



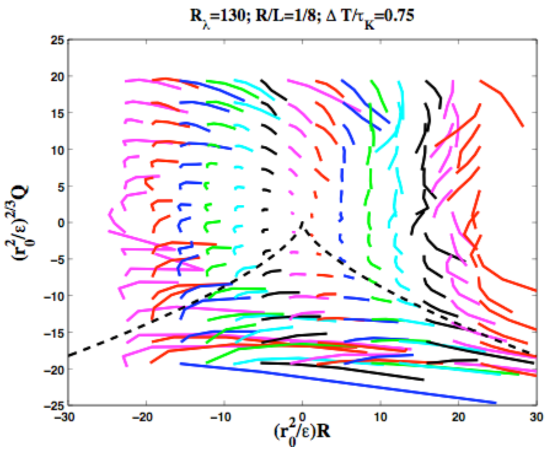
$r_0/L=1/2$



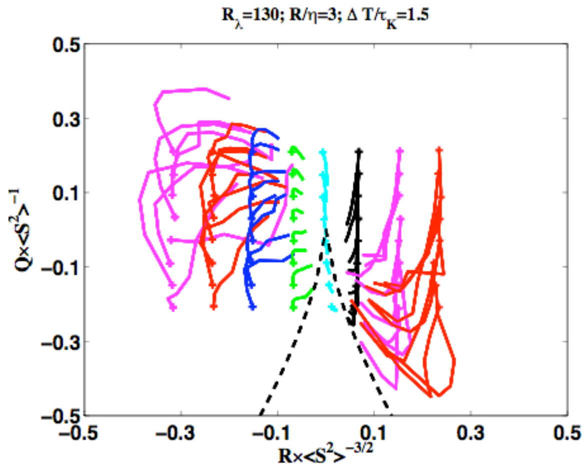
$r_0/L=1/8$

Trajectories similar while r_0 is in the inertial range !

Numerical results : Trajectories in the (R,Q) plane



$r_0/L=1/8$



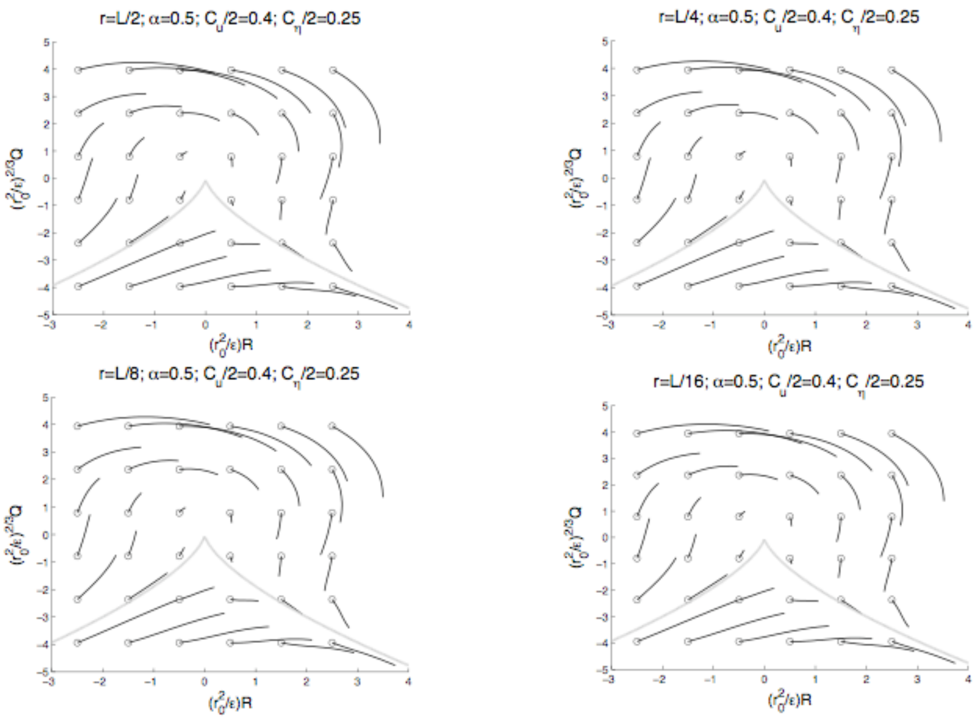
$r_0/L=1/32$

The character of the trajectories changes when r0 in the dissipative range (here, η/L ~ 100) !

n.b. : Difference with Chevillard + Meneveau (2006,2007)

Model predictions : Trajectories in the (R,Q) plane

r variable; α=0.5; C_u/2=0.4; C_η/2=0.25



- ✓***Little dependence on space*** r_0
- ✓***Qualitative similarities.***
- ✓***Quantitative differences (alignment properties not correctly taken into account)***

Conclusions and perspectives.

Conclusions and perspectives (1)

- Our work is based on a dynamical model of turbulent velocity fluctuations, that contains several key fluid mechanical ingredients.

- The model is formulated in terms of a **stochastic differential equations**, that depend on **3 dimensionless parameters**.

- The solutions have been obtained in the semiclassical limit, in two cases.

  - isotropic forcing : comparison with DNS results shows the important role of the **nonlinearity reduction** (role of the parameter α).

  - anisotropic forcing : difference in the properties of return to isotropy between ***vorticity*** dominated and ***strain*** dominated structures.

Conclusions and perspectives (2)

- Easy to study the influence of boundary conditions at large scales on small scales.
- In progress : development of an hybrid method that incorporates more precisely the fluctuations in the dynamics (… beyond the semiclassical approximation). Expected output : find out about the importance of the fluctuations as a function of the flow structures.
- Very recent development : new experimental results from the Göttingen (also Zürich, Risø and Lyon) group =>

  exciting new developments expected…

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