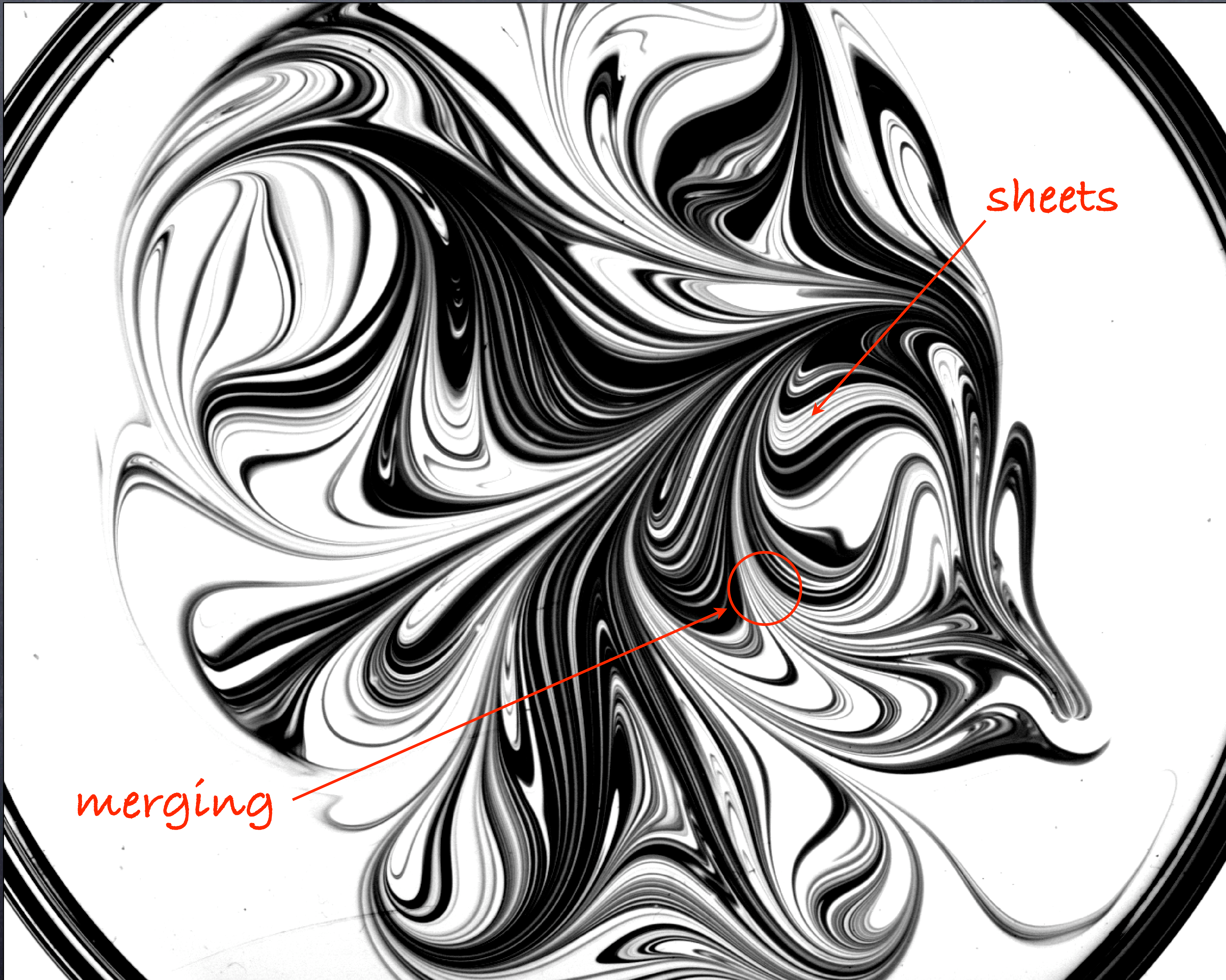
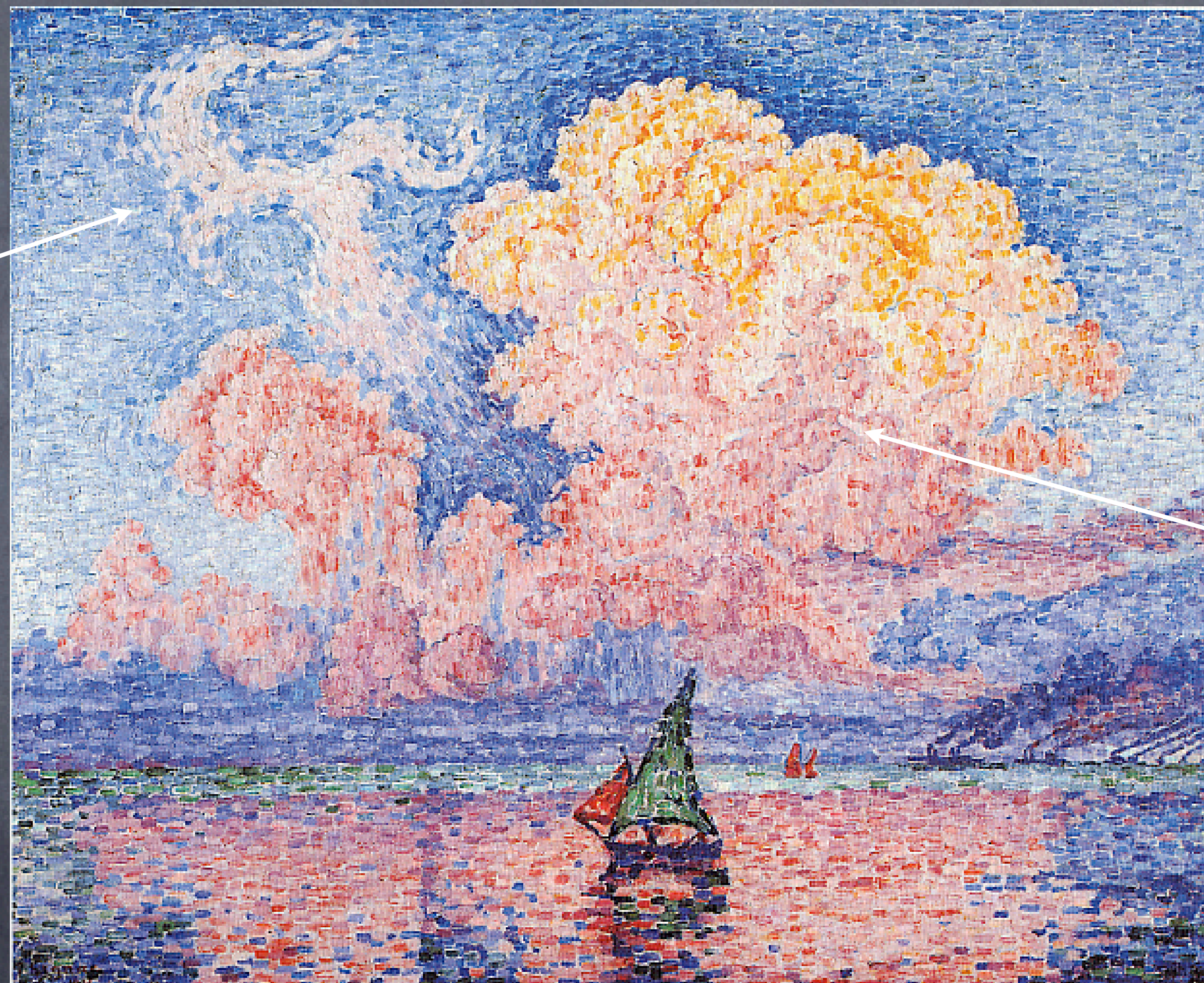
Mixing by Random Stirring


sheets
merging

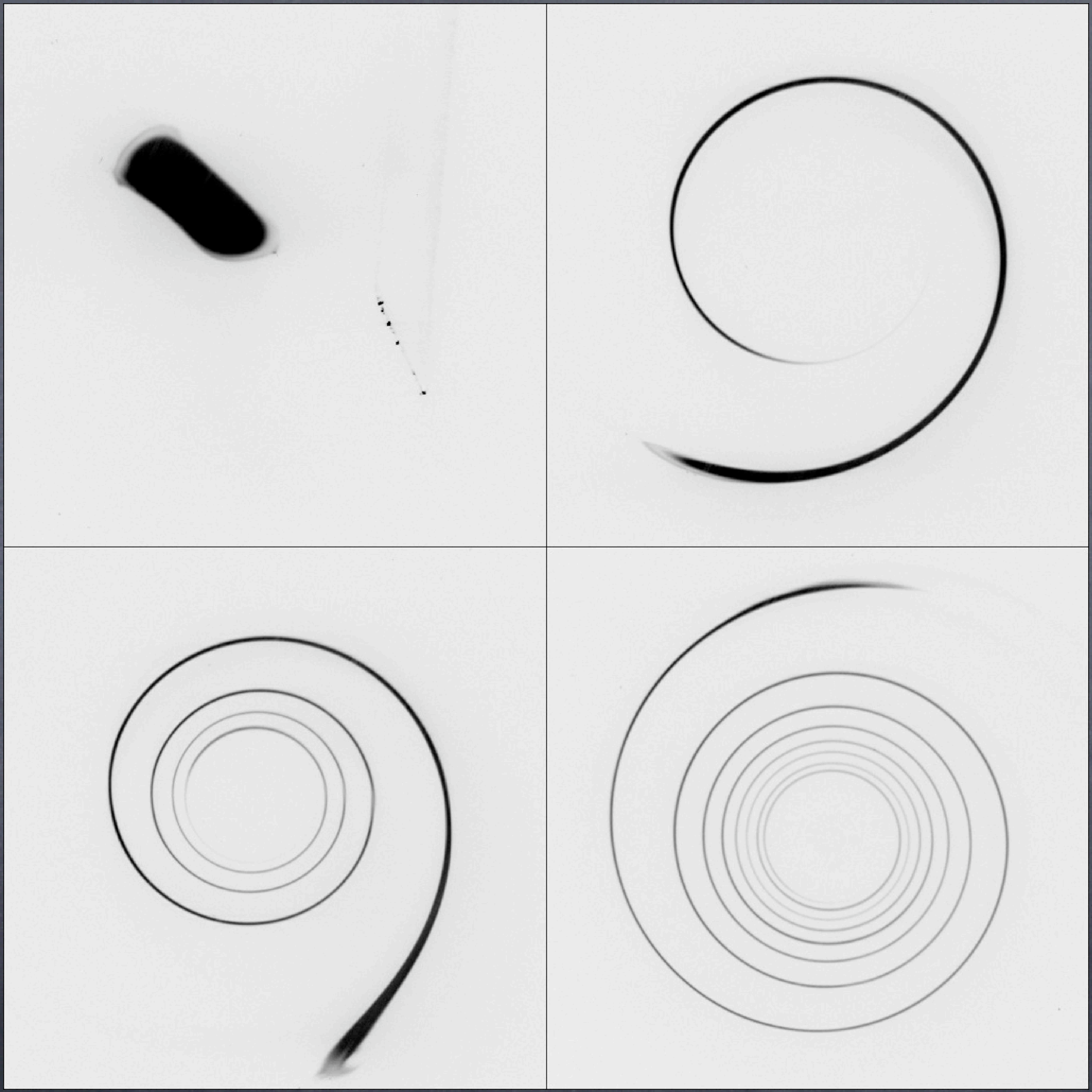
Question: $P(C) = ?$

Dispersing Mixture



Confined Mixture

P. Signac, 'Antibes, Le Nuage Rose' (1916)



Meunier & Villermaux, JFM 476 (2003)

Axisymmetric, diffusing vortex

$$\partial_t v_\theta = \nu \left\{ \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} \right\}$$

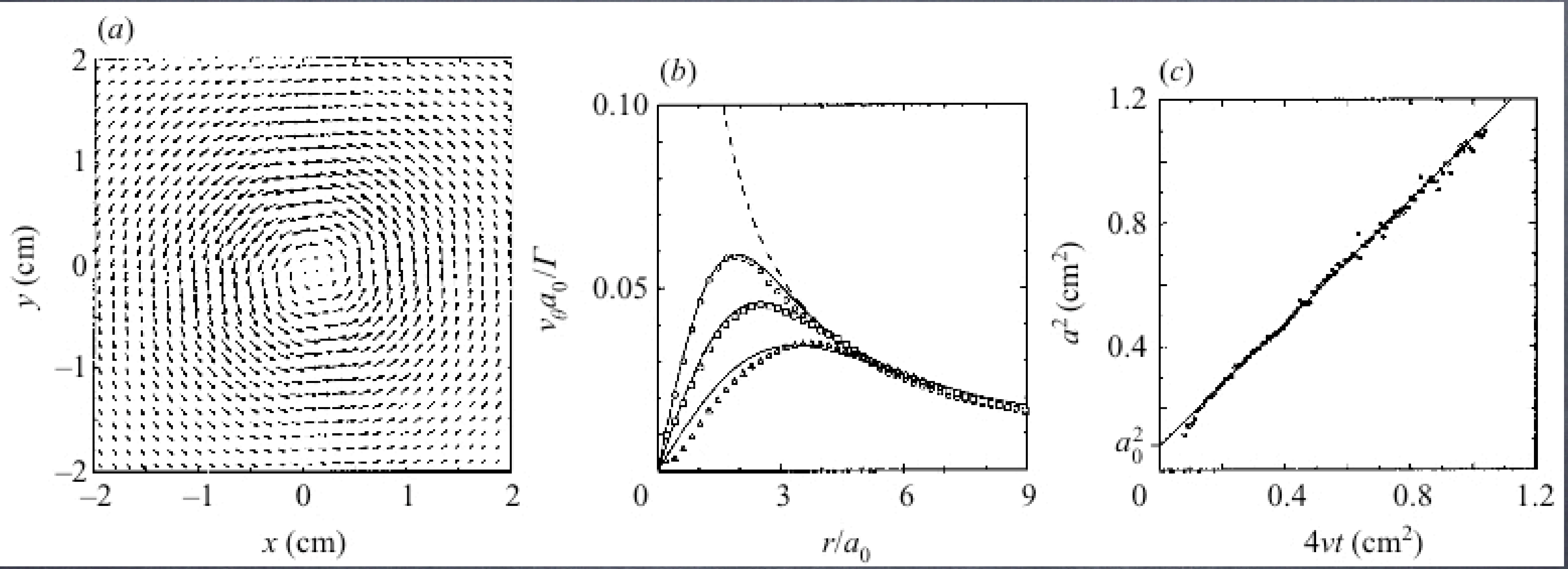
$$v_\theta = \frac{\Gamma}{2\pi r}\left(1 - e^{-r^2/a^2}\right)$$ Lamb-Oseen

$$a^2 = a_0^2 + 4\nu t$$

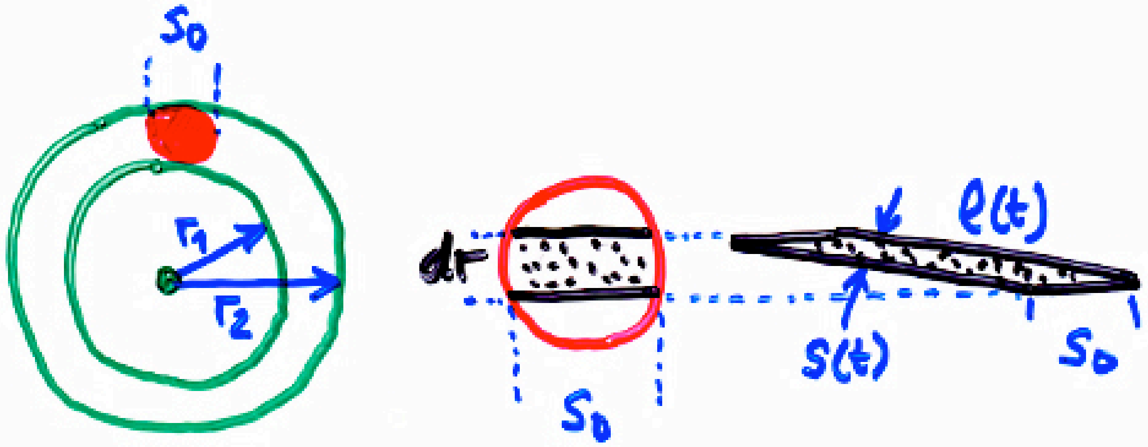
$$\Gamma = 2\pi a_0 v_\theta(a_0, t=0)$$

= circulation

Lamb-Oseen vortex



Scalar blob deformation



$r_1 \gg a(t)$: $v_\theta \approx \dfrac{\Gamma}{2\pi r}$

length: $d\ell(r,t) = \sqrt{dr^2 + (r\,d\theta)^2}$

$$= dr\sqrt{1 + \frac{\Gamma^2 t^2}{\pi^2 r^4}}$$

thickness: $s(t) = \dfrac{s_0\,dr}{d\ell(t)} = \dfrac{s_0}{\sqrt{1 + \frac{\Gamma^2 t^2}{\pi^2 r^4}}}$

Spiral total length:

$$\int_{r_1}^{r_2} d\ell(r,t) \sim t$$

Convection – diffusion

In the local frame of a lamellae :

substrate velocity : $u = -\frac{x}{s}\frac{ds}{dt}$

$v = \frac{y}{s}\frac{ds}{dt}$

Diffusion equation :

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right)$$

$$u\frac{\partial c}{\partial x} \Big/ v\frac{\partial c}{\partial y} = O\left(\frac{s}{\ell}\right) ; \quad \frac{\partial^2 c}{\partial x^2} \Big/ \frac{\partial^2 c}{\partial y^2} = O\left(\frac{s}{\ell}\right)^2$$

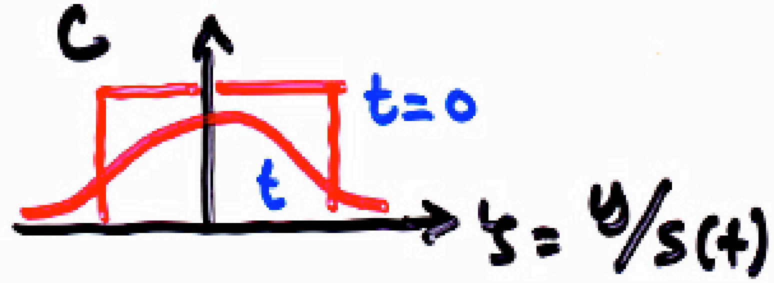
Thus : $$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial y} = D\frac{\partial^2 c}{\partial y^2}$$

With : $\xi = \frac{y}{s(t)}$; $\tau = D\int_0^t \frac{dt'}{s(t')^2}$

$$\boxed{\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial \xi^2}}$$

Solution:

$$c(\xi,\tau) = \frac{1}{2}\left\{\mathrm{erf}\left[\frac{\xi+1/2}{2\sqrt{\tau}}\right] - \mathrm{erf}\left[\frac{\xi-1/2}{2\sqrt{\tau}}\right]\right\}$$



Mixing time:

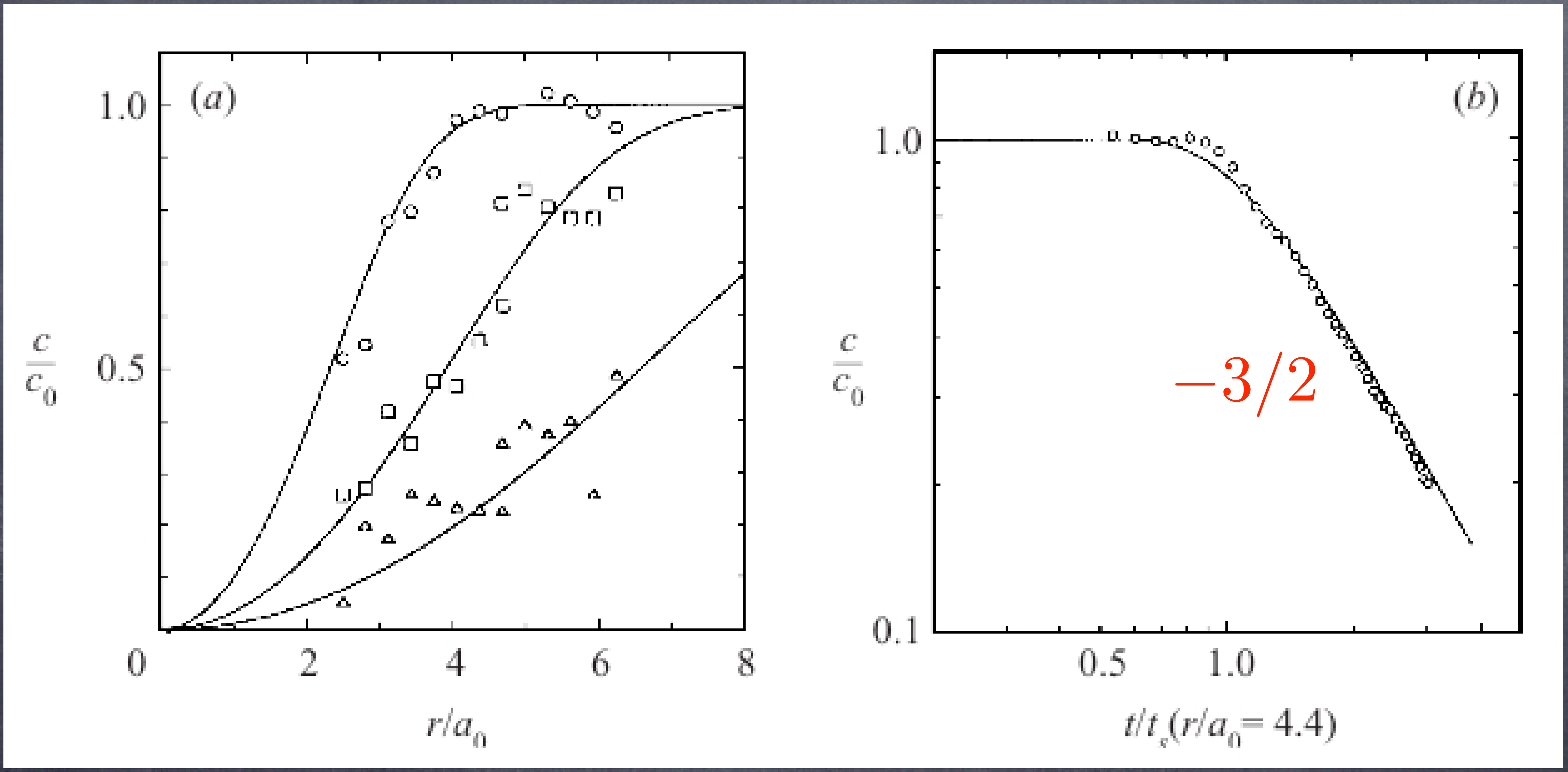
$$t_s(r) \cong \frac{r^2}{\Gamma'}\left(\frac{s_0}{r}\right)^{2/3}\left(\frac{\Gamma'}{D}\right)^{1/3}$$

For $t > t_s(r)$:

$$c(r,\xi) \approx \mathrm{erf}\left(\frac{1}{4\sqrt{\tau}}\right) e^{-\xi^2/2\tau}$$

With: $\tau(r,t) = \frac{Dt}{s_0^2} + \frac{D\Gamma'^2 t^3}{3\pi^2 r^4 s_0^2}$

$$c_{max}(r,t) \sim t^{-3/2} \quad \text{for } t > t_s$$



fixed times

fixed location

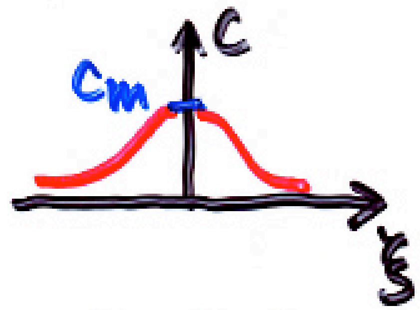
Spatial concentration field

$$c(r,\xi,t) \approx c_0\,\mathrm{erf}\left(\frac{1}{4\sqrt{\tau(r)}}\right)\exp\left(\frac{-6\,\xi^2}{1+24\tau(r)}\right)$$

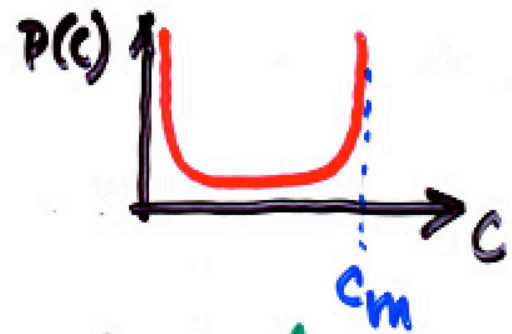
Concentration distribution

$$P(c) = 2s_0 \int_{c_M(r)>c} |\partial c/\partial \xi|^{-1}\,\mathrm{d}r/A \approx \frac{\sqrt{\tau+1/24}}{\partial c_M/\partial r}$$

(for details, see Meunier & Villermaux, JFM 476, 2003) with: $\left|\frac{\partial c}{\partial \xi}\right| = \sqrt{\frac{2}{\tau(r)}} \cdot c \cdot \sqrt{\ln\left(\frac{c_m(r)}{c}\right)}$

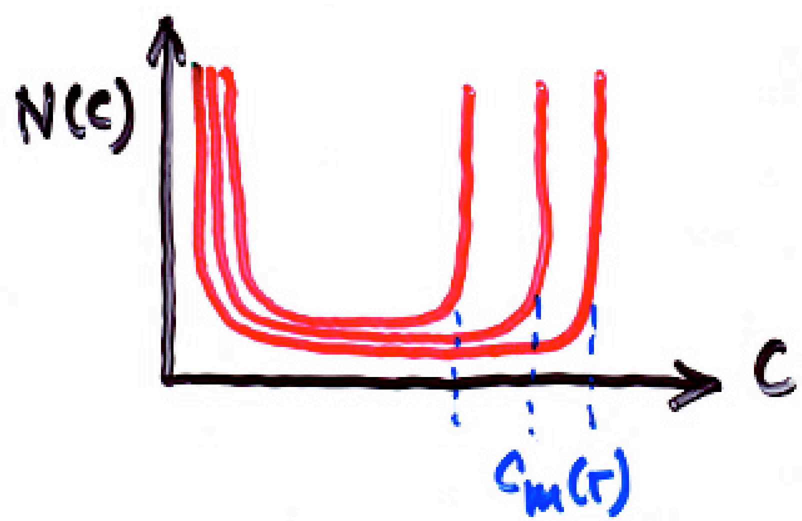


Spatial profile

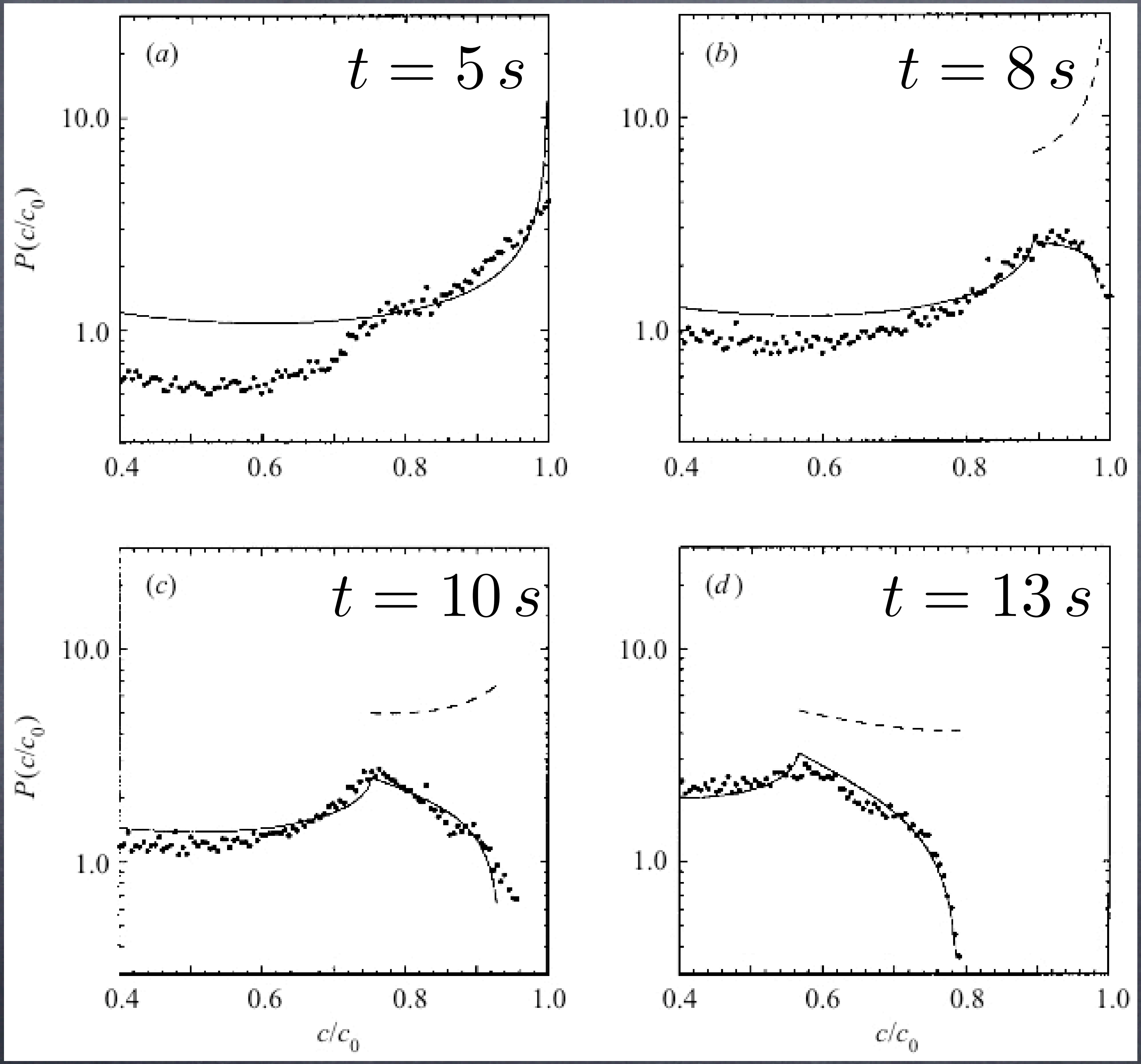


Concentration distribution

$$N(c)\,dc = \int d\mu\, d\eta = \int_{r(c)}^{r_2} \frac{dr\,dc}{\partial c/\partial \xi}$$

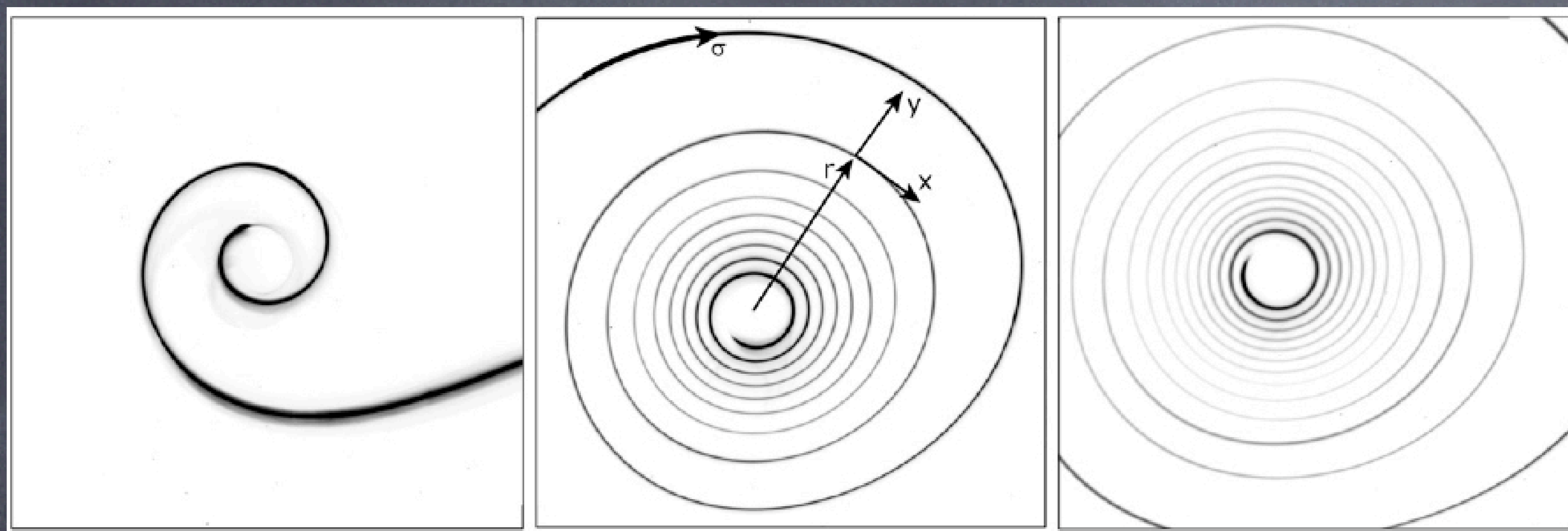


Superposition of U-shaped elementary distributions

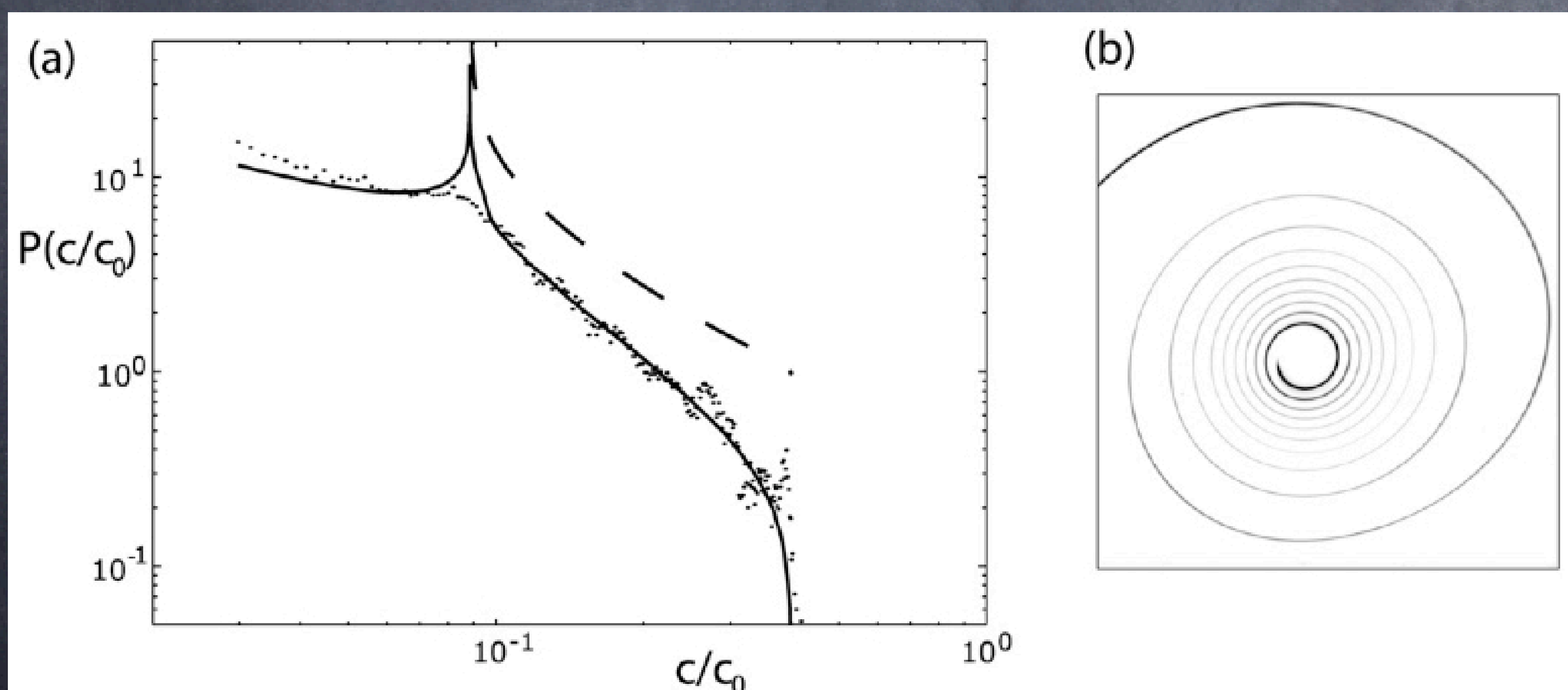


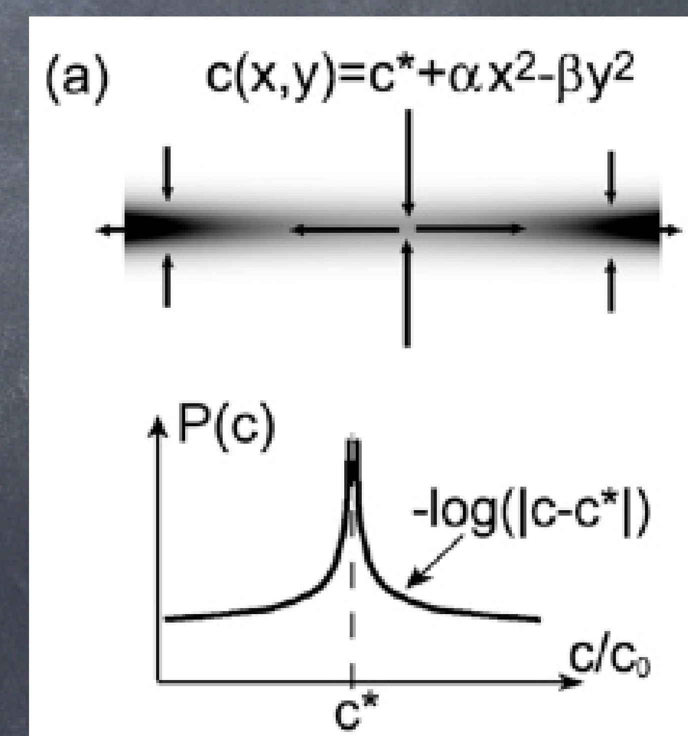
(a)
t = 5 s
(b)
t = 8 s
(c)
t = 10 s
(d)
t = 13 s
P(c/c0)
10.0
1.0
0.4
0.6
0.8
1.0
c/c0

van Hove Singularities

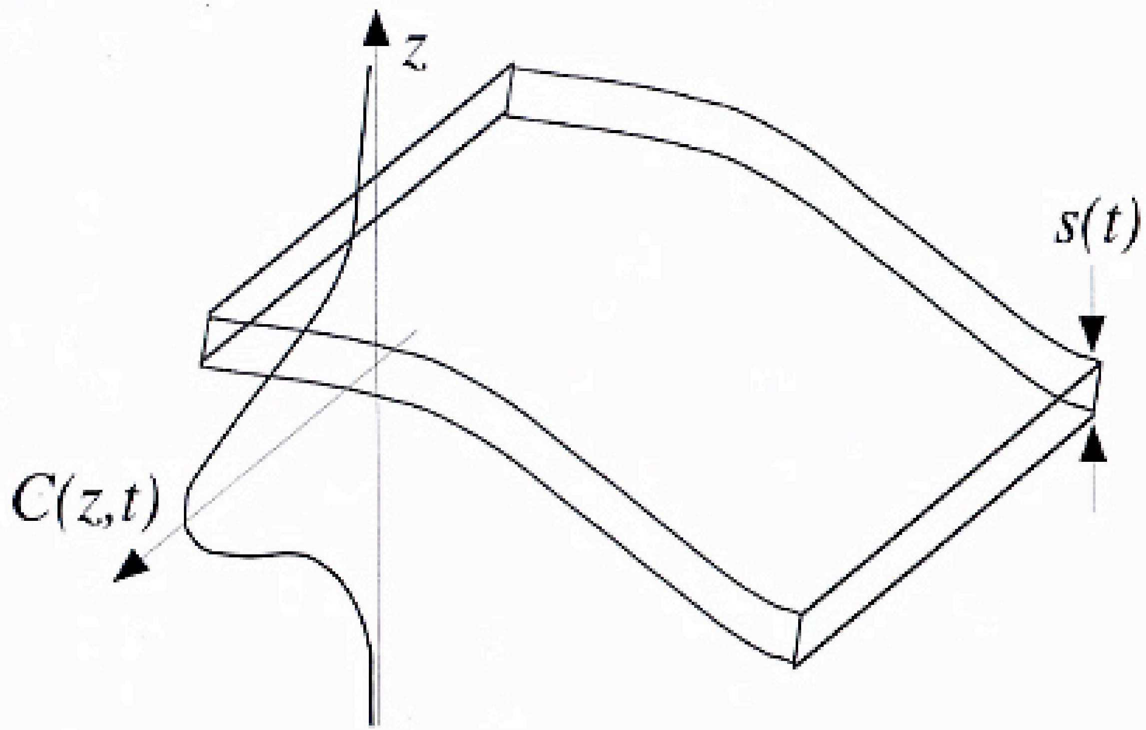


spatial saddle point in c(x,y)





Meunier & Villermaux, CRAS 335 (2007)



$$\frac{\partial c}{\partial t} + \frac{d \ln s}{dt} \cdot \frac{\partial c}{\partial z} = D \frac{\partial^2 c}{\partial z^2}$$

Mixing time : $\frac{D}{s^2} \sim \frac{1}{s}\frac{ds}{dt}$

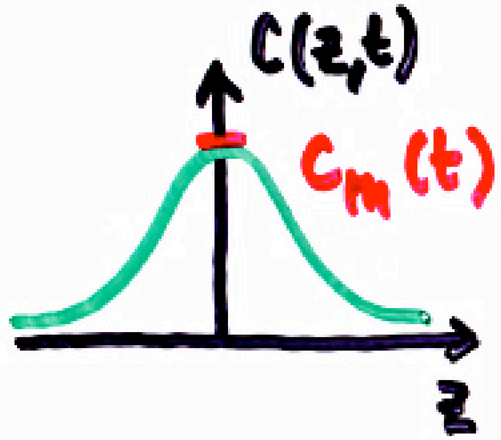
Let : $s = L(\gamma t)^{-\alpha}$, then $\gamma t_s \sim \left(\frac{\gamma L^2}{D}\right)^{\frac{1}{1+2\alpha}}$

$Pe = \frac{\gamma L^2}{D} > 1$

- $\alpha = 1 \rightarrow \frac{1}{1+2\alpha} = \frac{1}{3}$
- $\alpha = 2 \rightarrow$ " $= \frac{1}{5}$

Let : $s = L e^{-\gamma t}$, then $\gamma t_s \sim \frac{1}{2} \ln\left(\frac{\gamma L^2}{D}\right)$

After the mixing time

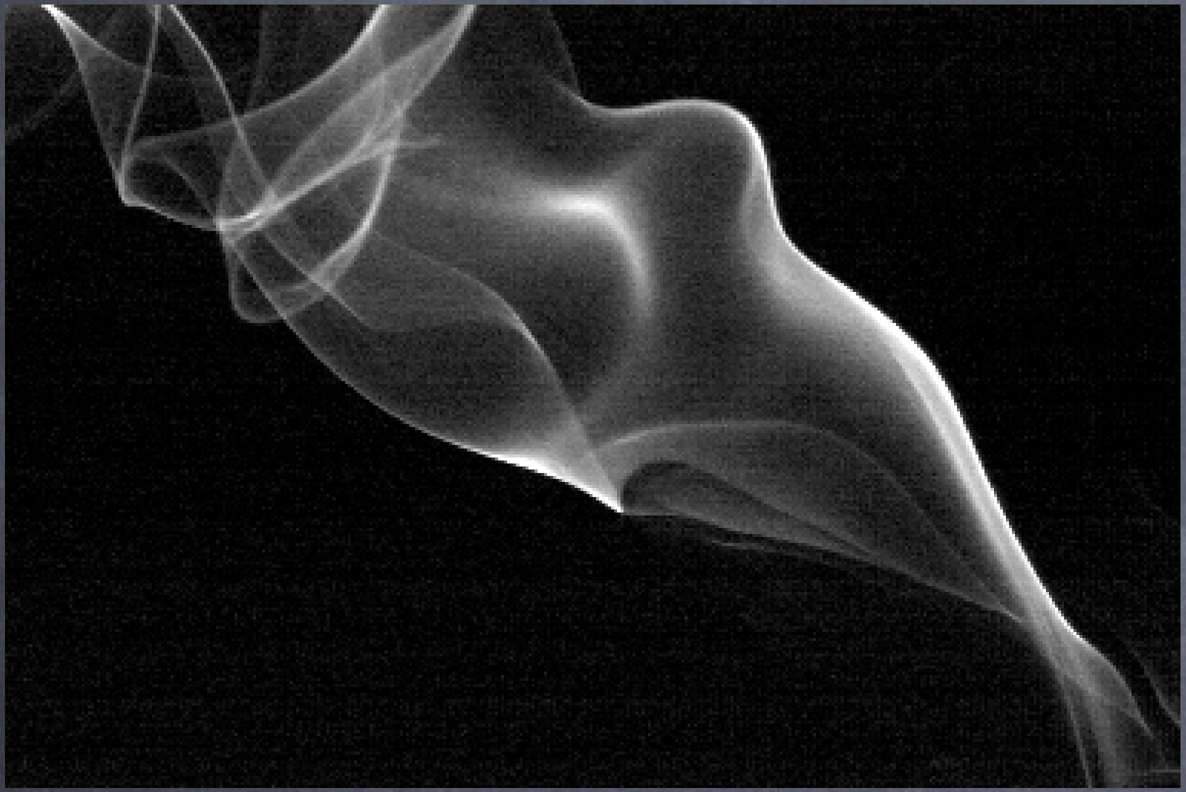


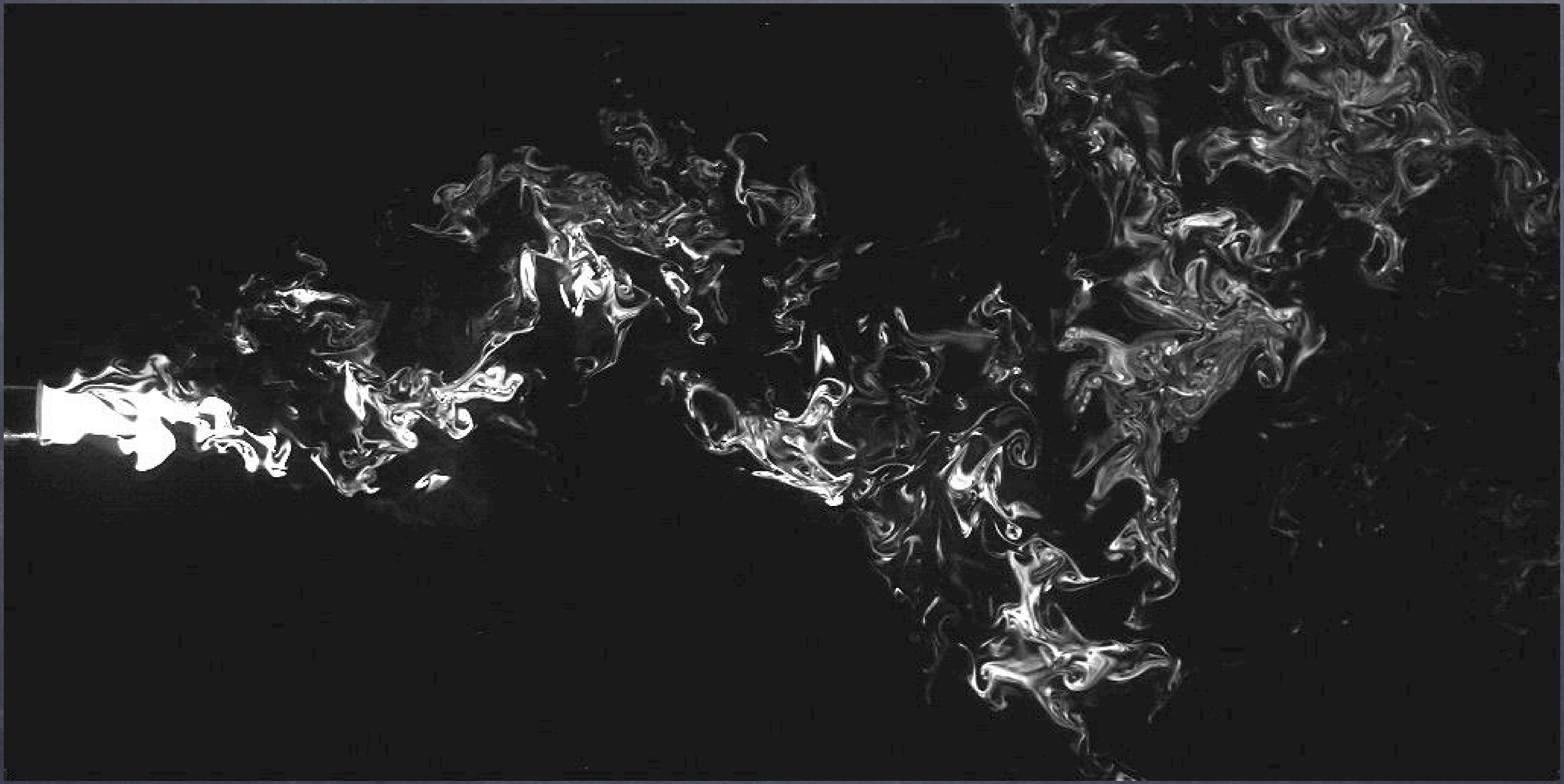
Maximal concentration:

$$c_m(t) \cdot \frac{\sqrt{Dt}}{S(t)} = \text{const.}$$

Thus: $c_m(t) \sim (\gamma t)^{-\frac{1}{2}-\alpha}$

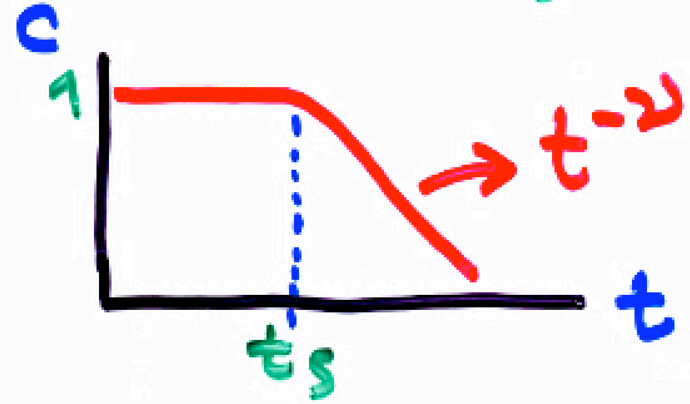
- $\alpha = 1 \rightarrow c_m(t) \sim (\gamma t)^{-3/2}$
- $\alpha = 2 \rightarrow c_m(t) \sim (\gamma t)^{-5/2}$



- Concentration

$$c = \left(1 + \frac{t}{t_s}\right)^{-\nu} ; \quad \nu = \alpha + \frac{1}{2}$$



- Distribution of elongations

$$P(\ell, t)$$

Liouville: $P(\ell + \delta\ell, t + \delta t) = P(\ell, t)$

$$\partial_t P = - \partial_\ell \left\{ \frac{\delta \ell}{\delta t} \cdot P \right\}$$

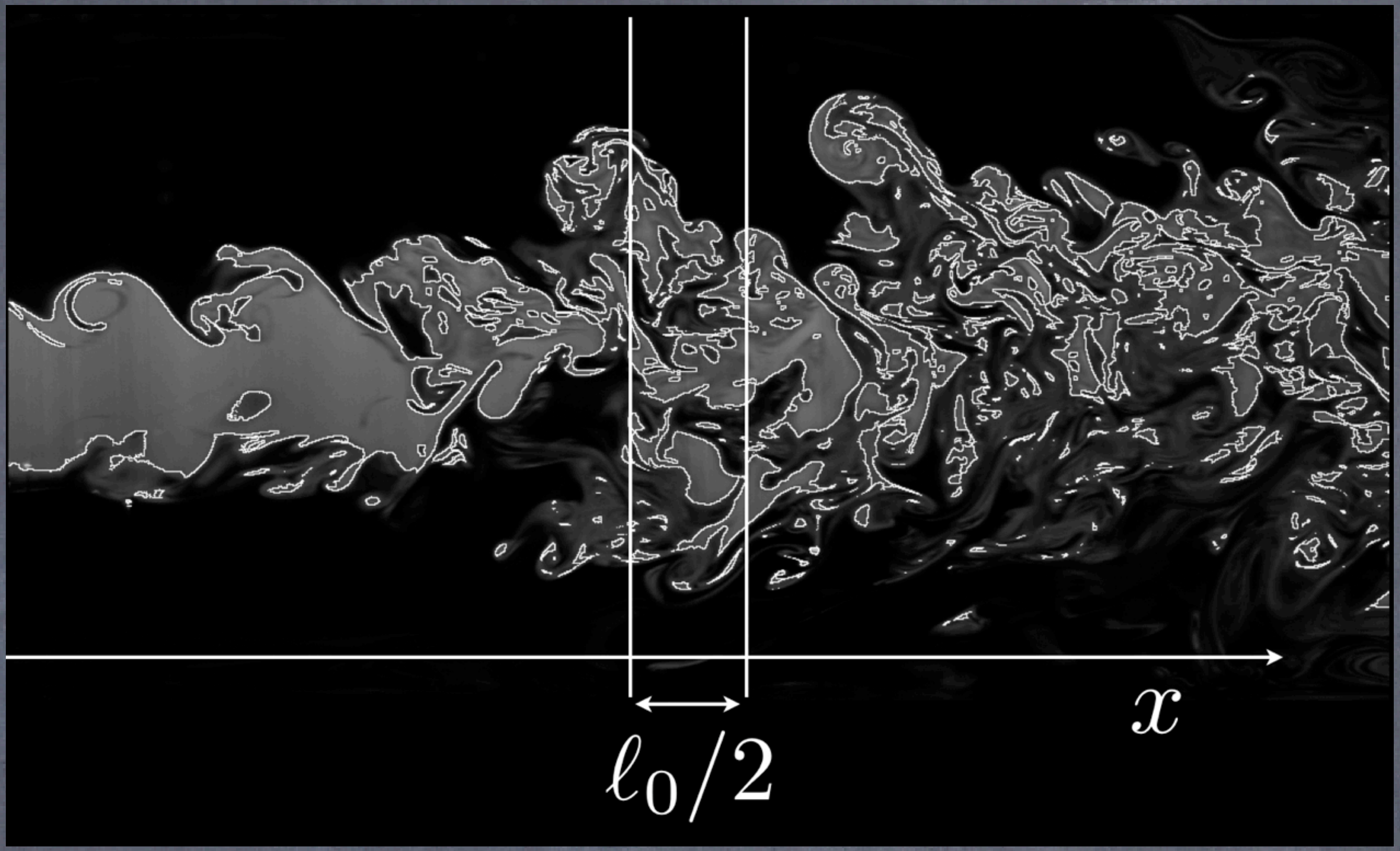
if: $\frac{\delta \ell}{\delta t} = \frac{\ell}{t}$ (linear flows)

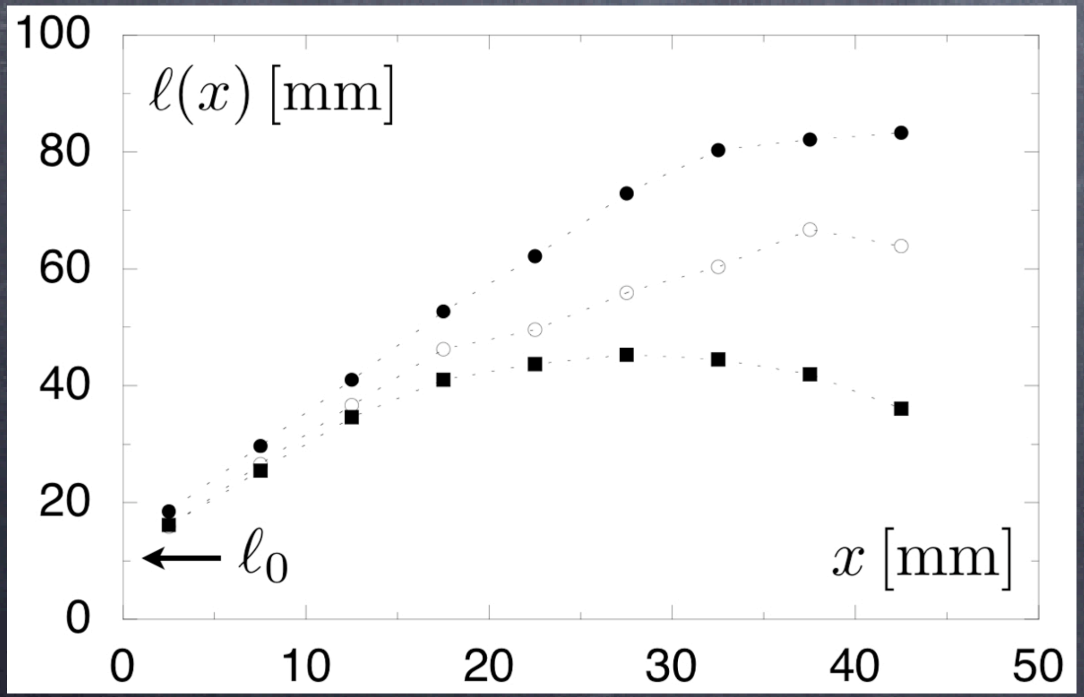
then: $\partial_t P = - \partial_\ell \left\{ \frac{\ell}{t} P \right\}$

with: $P(\ell, t=0) = \delta(\ell = 0)$

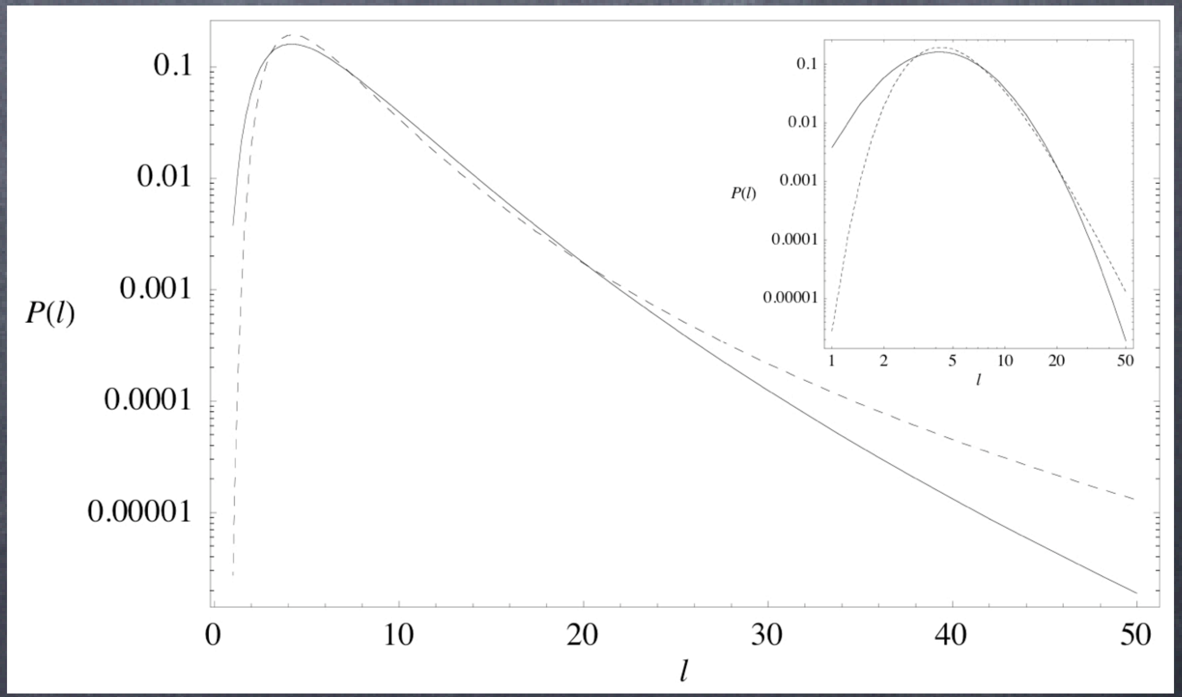
$$\rightarrow \quad P(\ell, t) = \frac{ut}{\ell^2} \exp\left\{ - \frac{ut}{\ell} \right\}$$

Contours lengths



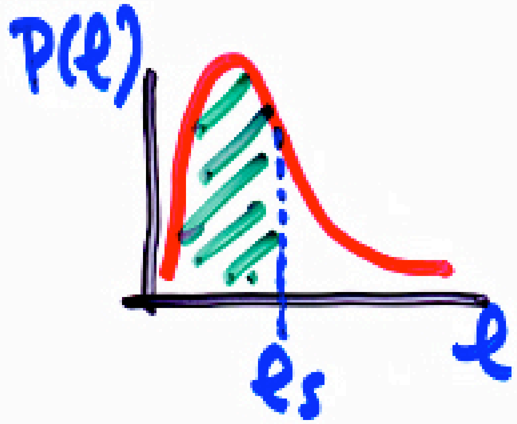


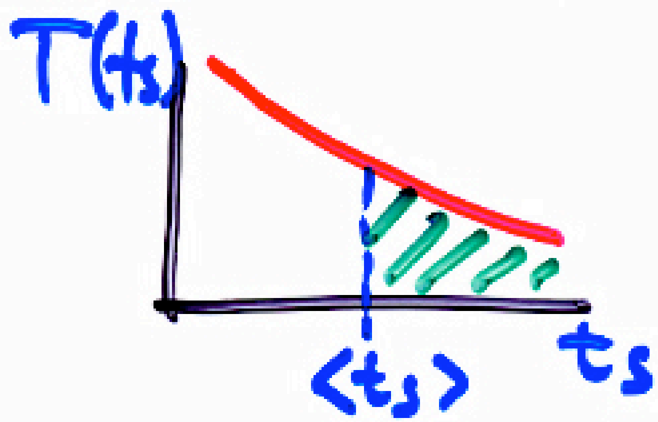
mean



distribution P(l)

- Distribution of mixing times





$$\int_{t_s}^{\infty} T(t_s)\,dt_s = \int_0^{\ell_s} P(\ell,t)\,d\ell$$

$$T(t_s) = -\partial_{t_s}\int_0^{\ell_s} P(\ell,t)\,d\ell$$

$$\rightarrow \quad T(t_s) = \frac{1}{\langle t_s\rangle}\exp\left\{-\frac{t_s}{\langle t_s\rangle}\right\}$$

also: Shraiman & Siggia, PRE (1994)

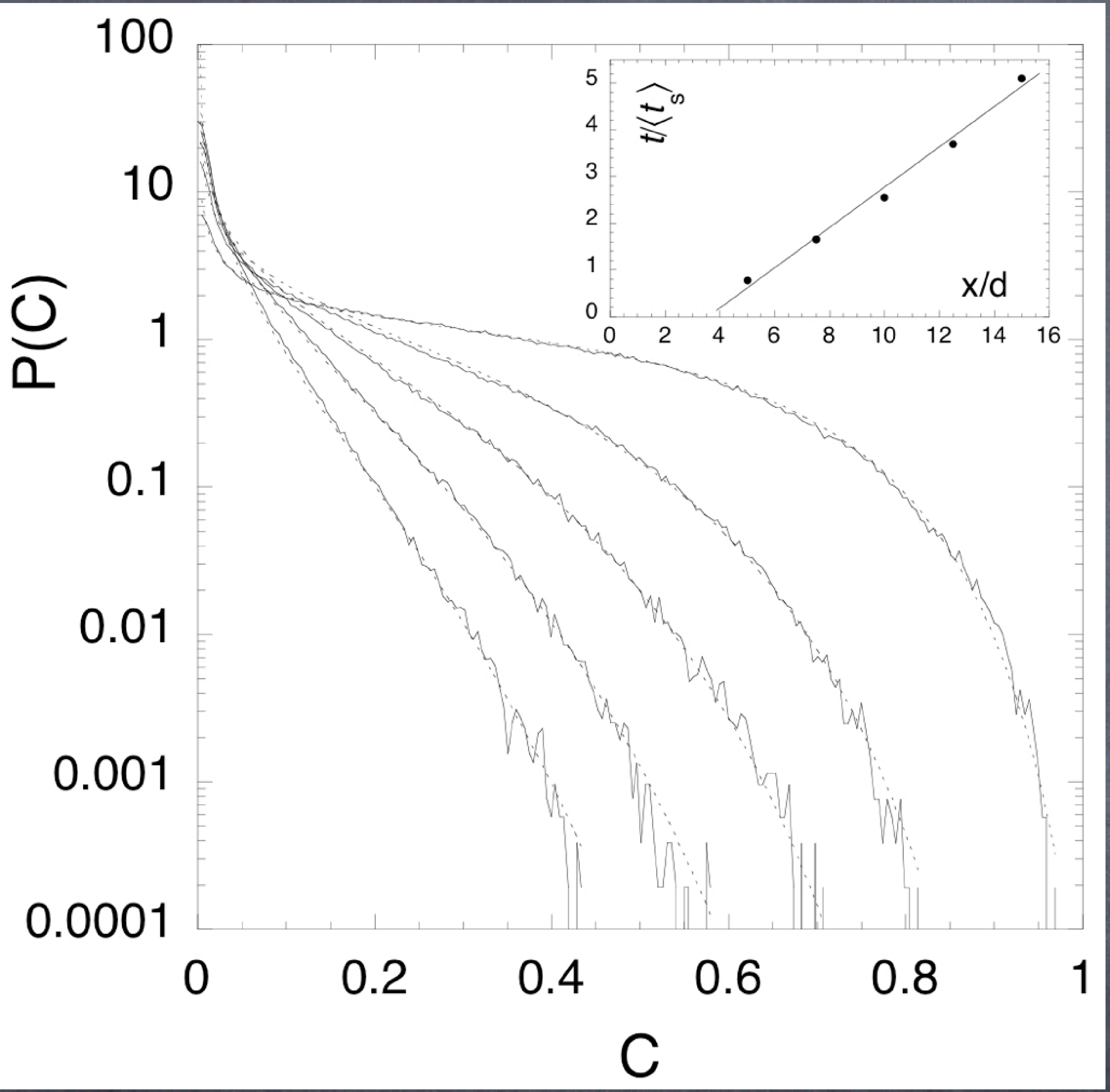
- Distribution of concentration $P(c,t)$

$$P(c,t)\,dc = T(t_s)\,dt_s$$

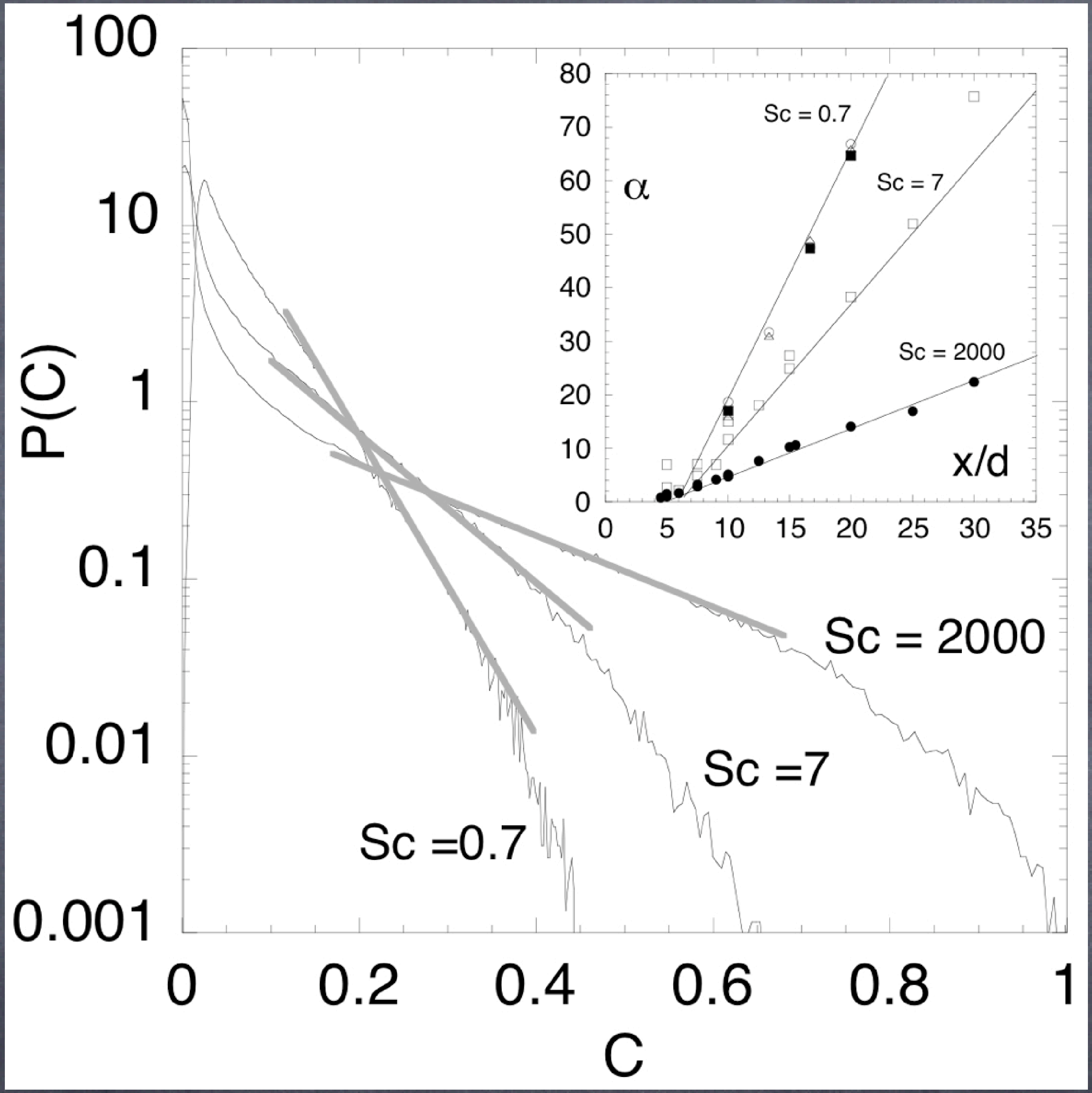
with $c = \left(1+\frac{t}{t_s}\right)^{-\nu}$

thus:

$$\begin{cases} p(c,t) = \dfrac{\tilde{t}}{\nu} \dfrac{c^{-\frac{1+\nu}{\nu}}}{\left(c^{-\frac{1}{\nu}}-1\right)^2} \exp\left\{-\dfrac{\tilde{t}}{c^{-\frac{1}{\nu}}-1}\right\} \\ \tilde{t} = \dfrac{t}{\langle t_s \rangle} \end{cases}$$



increasing times, Sc=7



fixed time, various scalars

interacting plumes

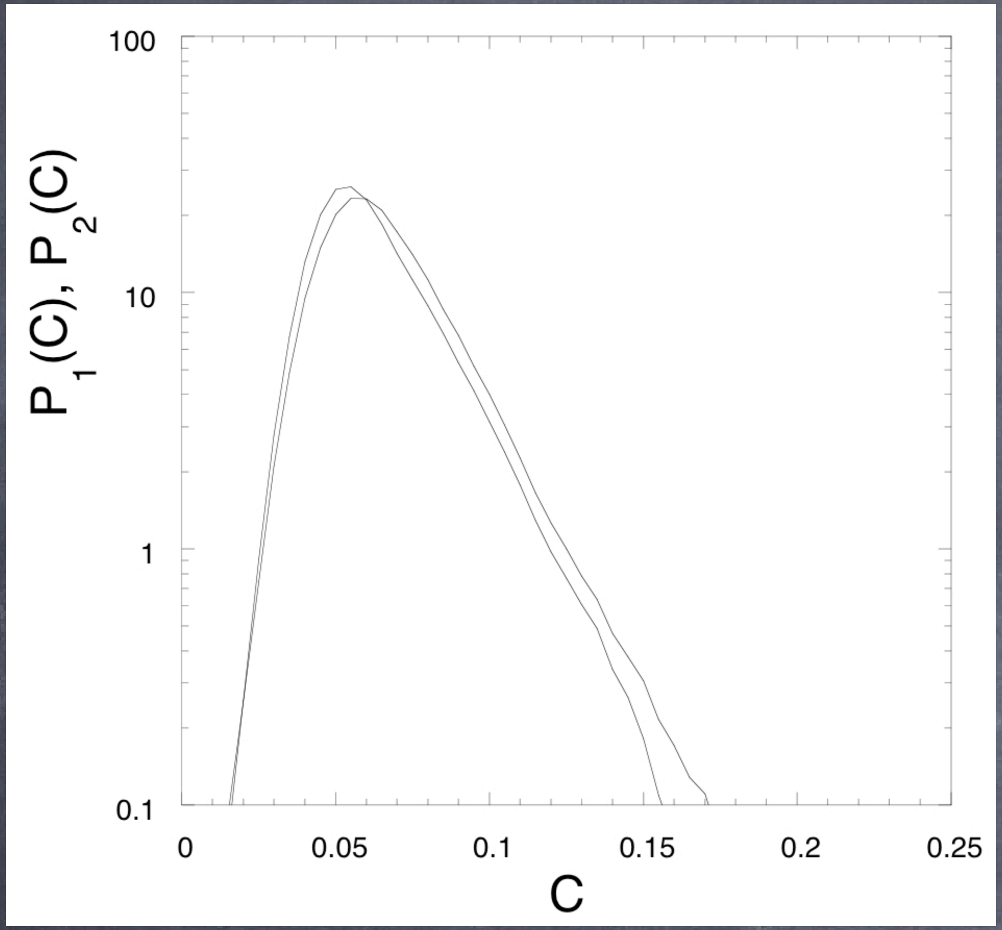
Composition rule

Composition rule

source 1, then source 2

Composition rule

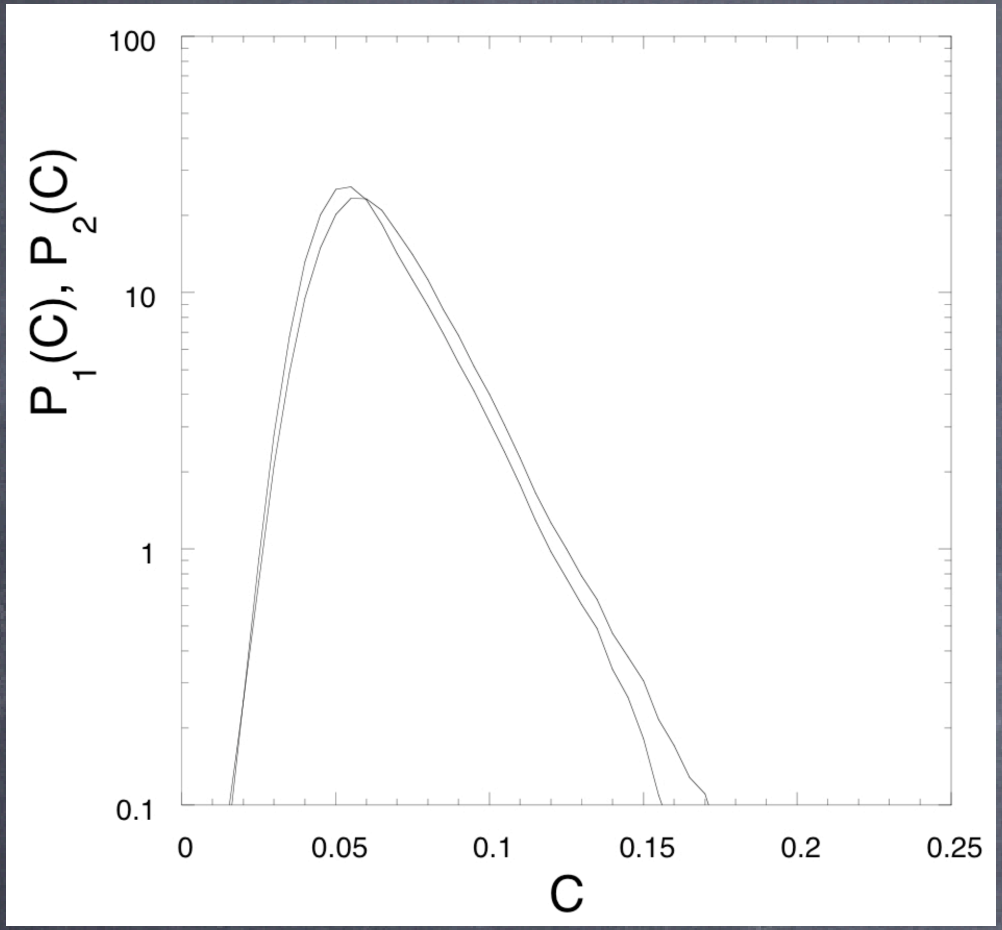
source 1, then source 2



Composition rule

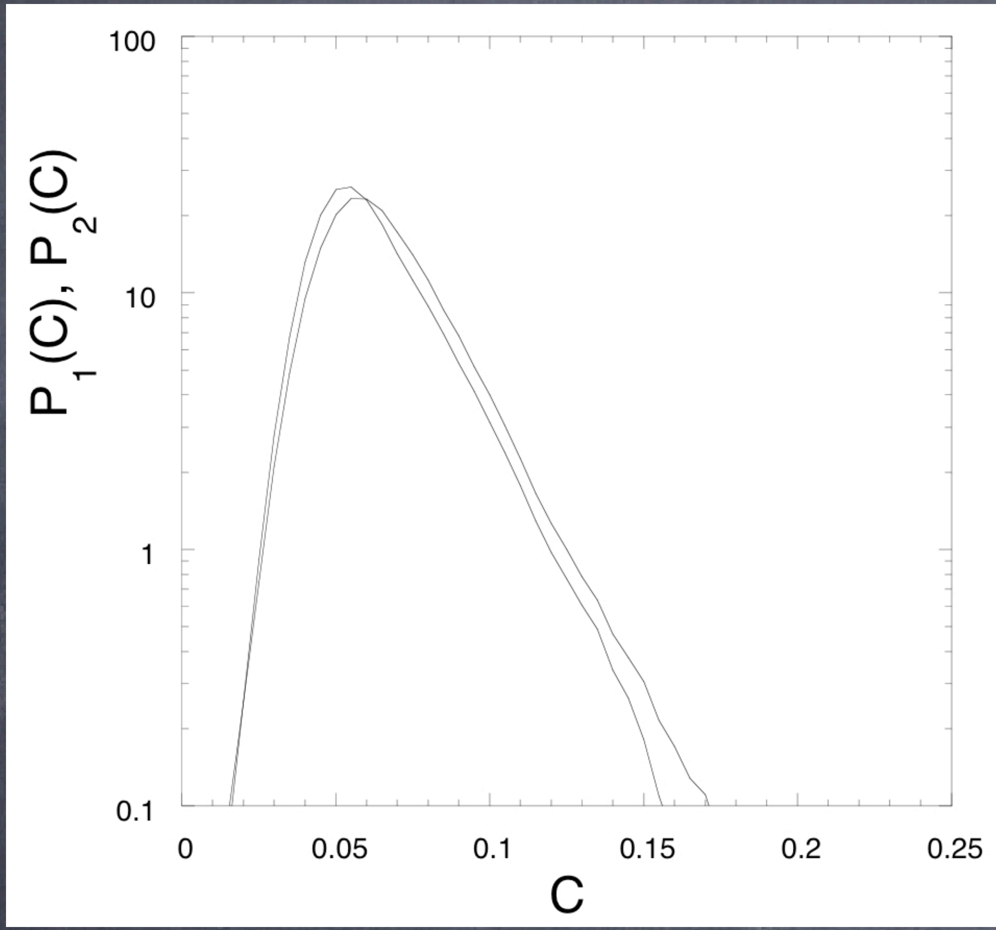
source 1, then source 2

source 1+source 2

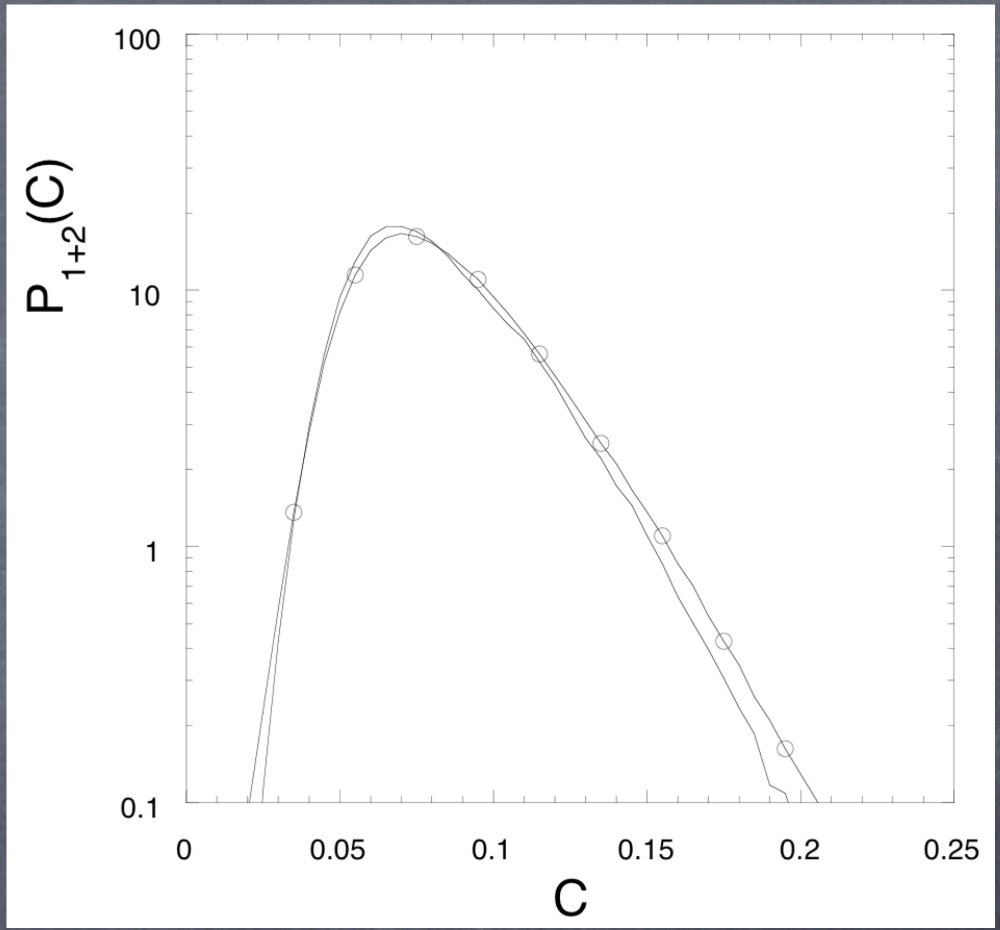


Composition rule

source 1, then source 2

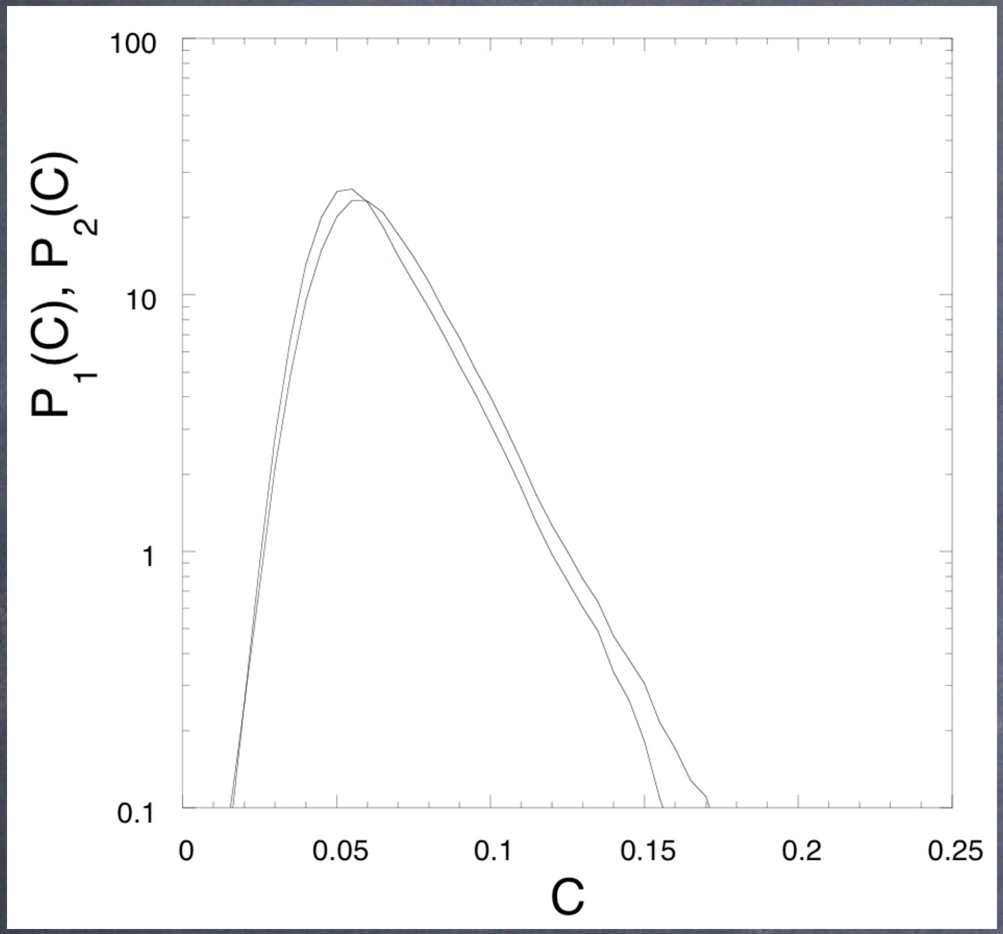


source 1+source 2

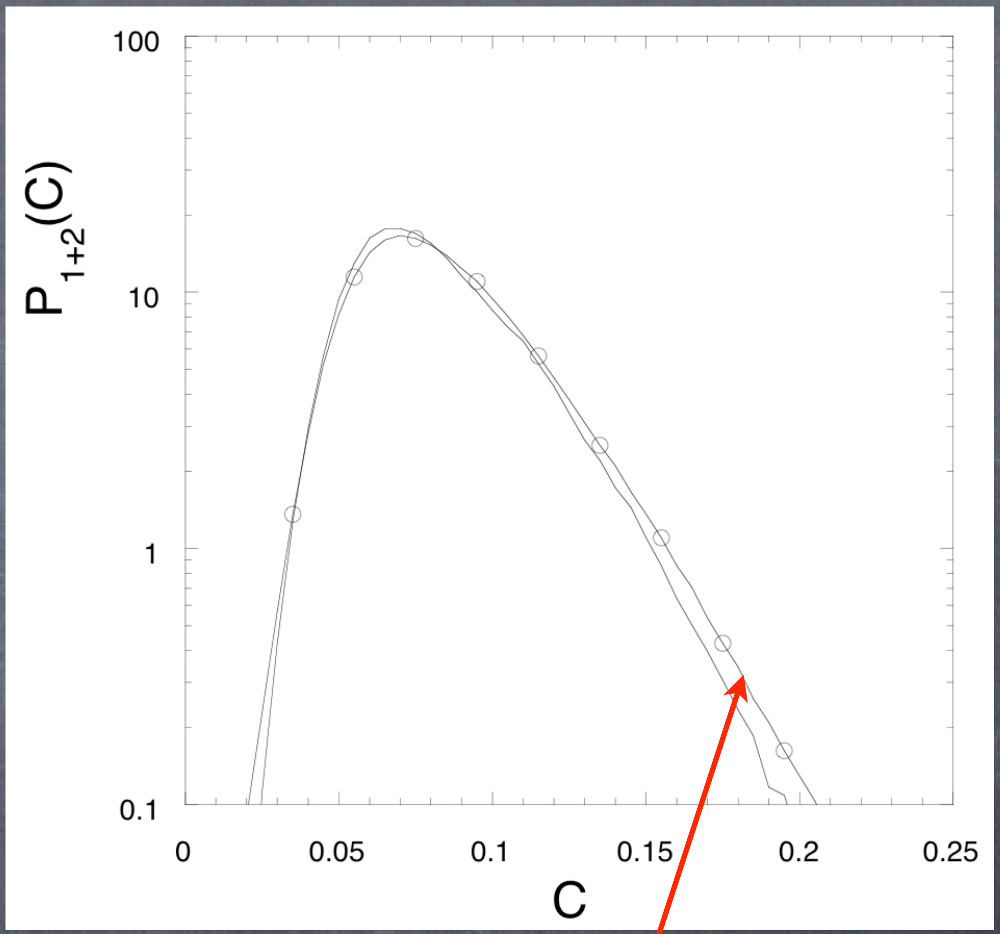


Composition rule

source 1, then source 2



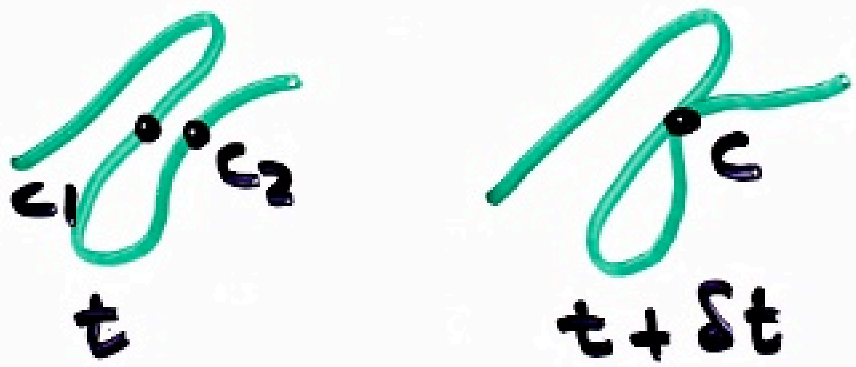
source 1+source 2



$$P(C) = \int_{C=C_1+C_2} P(C_1)P(C_2)\mathrm{d}C_2$$

Microscopic interaction

Self convolution processes



$c = c_1 + c_2$
(Fourier)

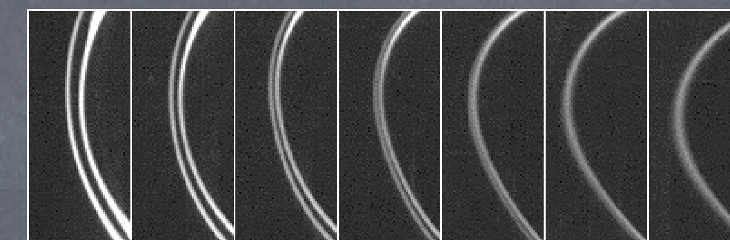
$$p(c,t+\delta t) = \int_{c=c_1+c_2} p(c_1,t)\, p(c_2,t)\, dc_2$$

Convolution

A useful tool: $\tilde{p}(s,t) = \int_0^\infty p(c,t)\, e^{-sc}\, dc$

Laplace transform

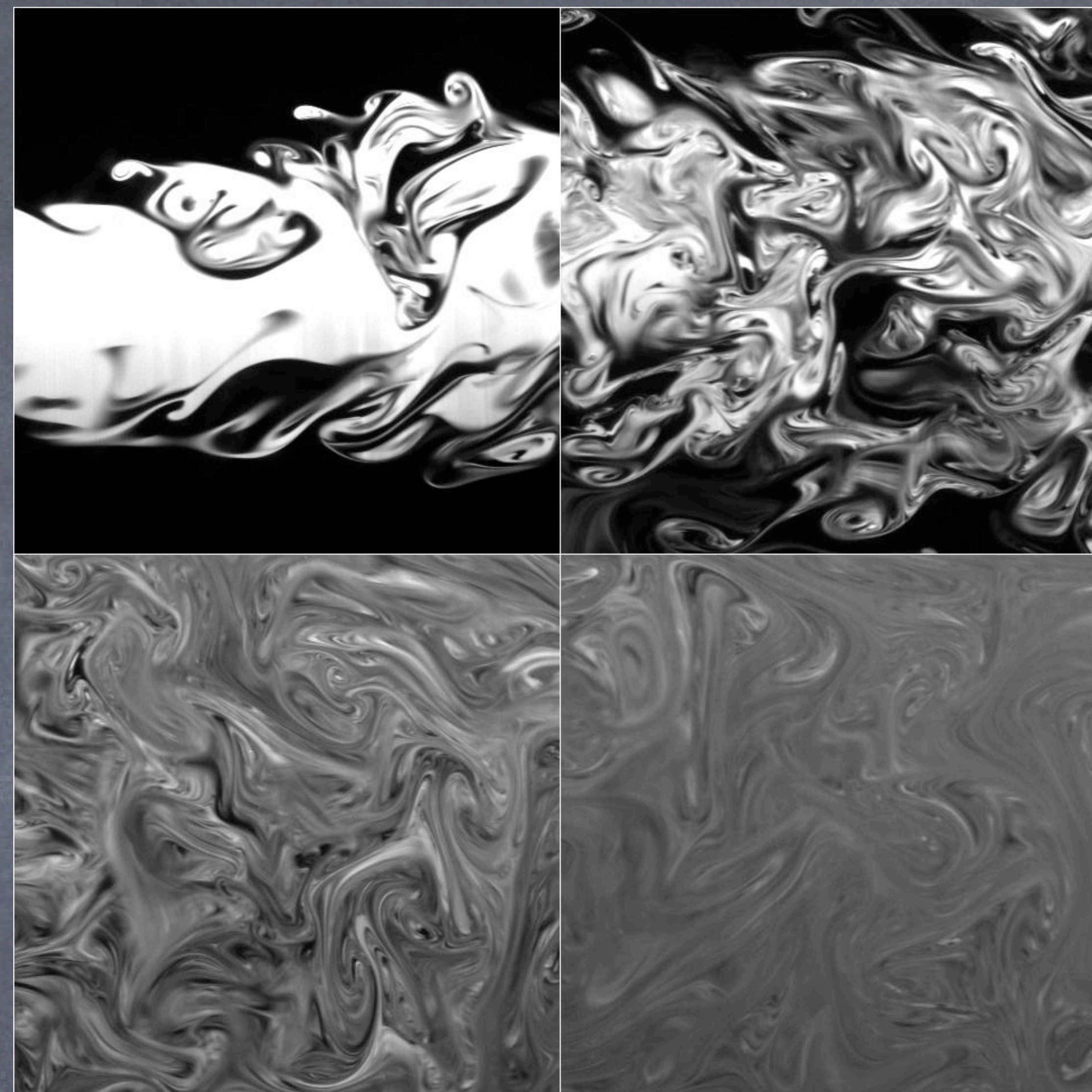
$$p(c,t) \otimes p(c,t) \longleftrightarrow \tilde{p}(s,t)^2$$



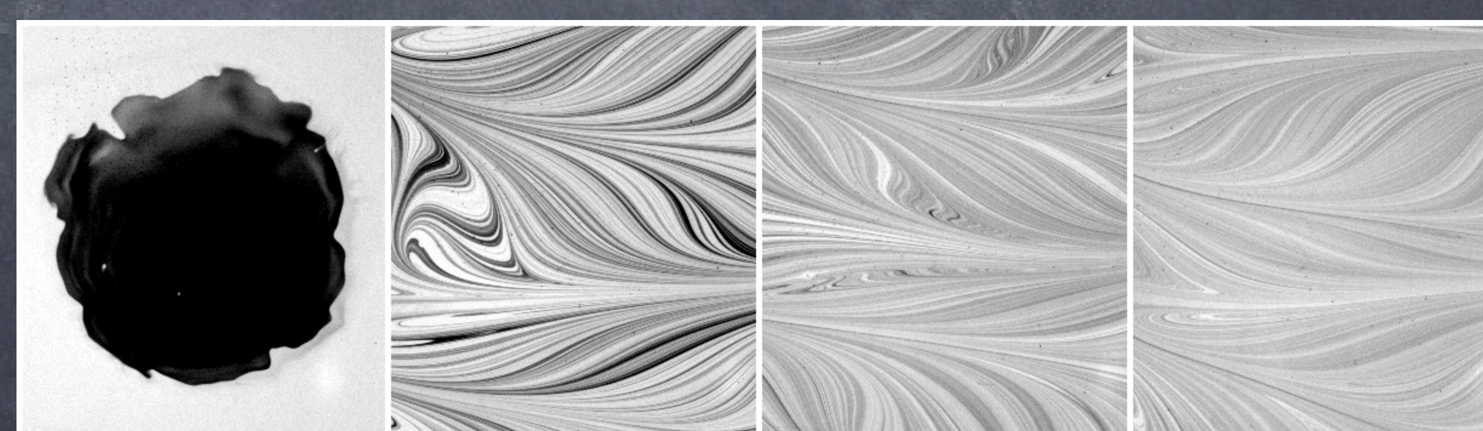
For real !

Confined mixtures

turbulent,
self stirred



laminar,
externally stirred



Possible scenarii:

(1) $\tilde{p}(s,t+\delta t) = \varepsilon\, \tilde{p}(s,t)^2 + (1-\varepsilon)\, \tilde{p}(s,t)$

with $\varepsilon = r\,\delta t$

$\partial_t \tilde{p} = r(-\tilde{p} + \tilde{p}^2)$

Smoluschowski 1917

(2) $\tilde{p}(s,t+\delta t) = \tilde{p}^{1+\varepsilon}$

$= \tilde{p} \exp\{\varepsilon \ln \tilde{p}\}$

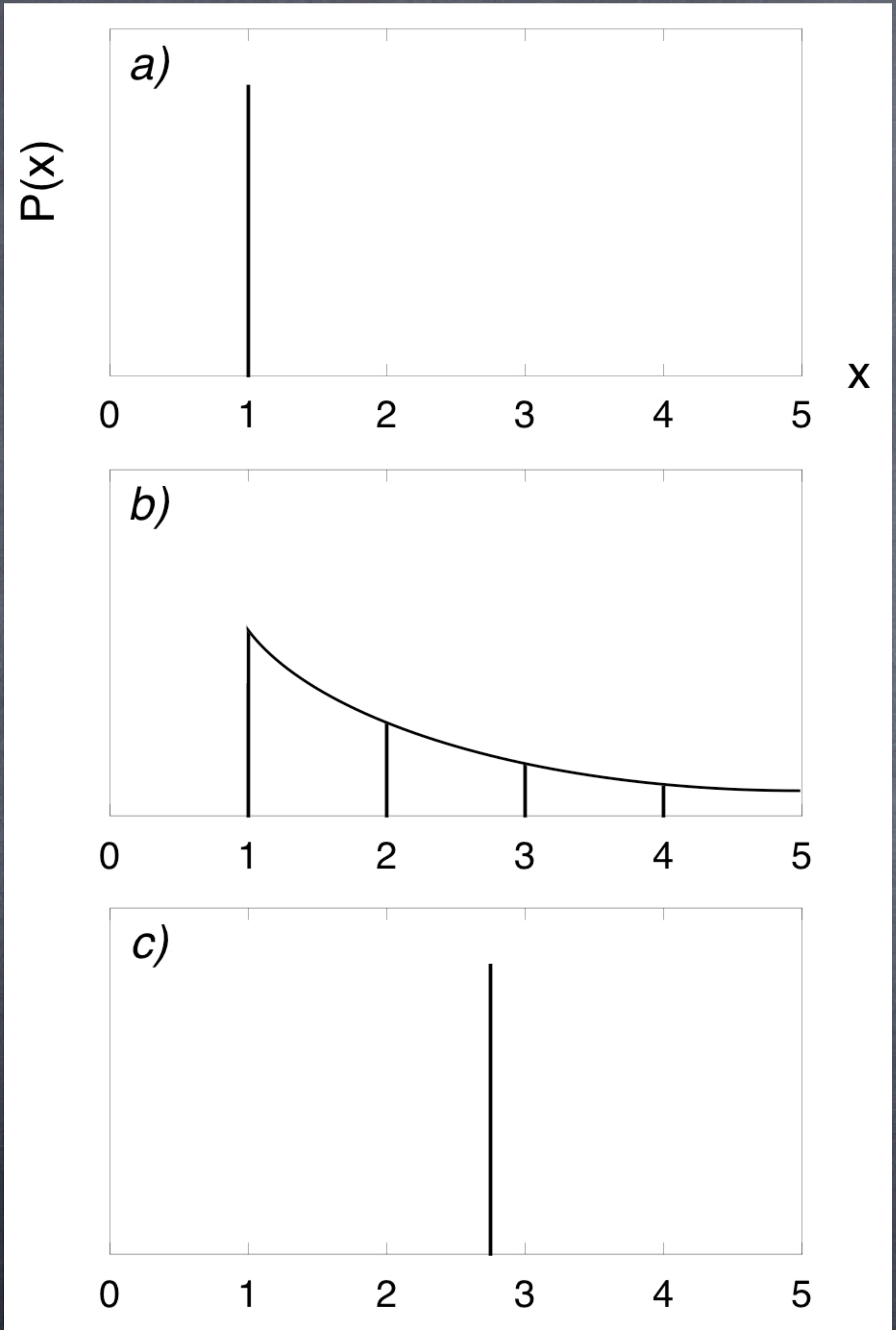
with $\varepsilon \to 0$: $\partial_t \tilde{p} = r\, \tilde{p} \ln \tilde{p}$

(3) General:

$\partial_t \tilde{p} = r n \{-\tilde{p} + \tilde{p}^{1+\frac{1}{n}}\}$

$n = 1 \longrightarrow (1)$

$n = \infty \longrightarrow (2)$



Initial condition

$$P(x, t=0) = \delta(x-1)$$

$$\partial_t \tilde{P} = -\tilde{P} + \tilde{P}^2$$

$$P(x,t) = e^{-x/e^t}/e^t$$

$$\partial_t \tilde{P} = \tilde{P} \ln \tilde{P}$$

$$P(x,t) = \delta(x - e^t)$$

Plus global shift by stretching:

$$\partial_t p = -\frac{\partial}{\partial c}\left\{ \frac{\delta c}{\delta t}\, p \right\}$$

$$\frac{\delta c}{\delta t} = -\gamma(t)\, c \; ; \; \gamma(t) = \frac{\alpha + \frac{1}{2}}{t}$$

finaly:

$$\partial_t \tilde{p} = -\gamma s\, \partial_s \tilde{p} + r n \left\{ -\tilde{p} + \tilde{p}^{1+\frac{1}{n}} \right\}$$

Solution:

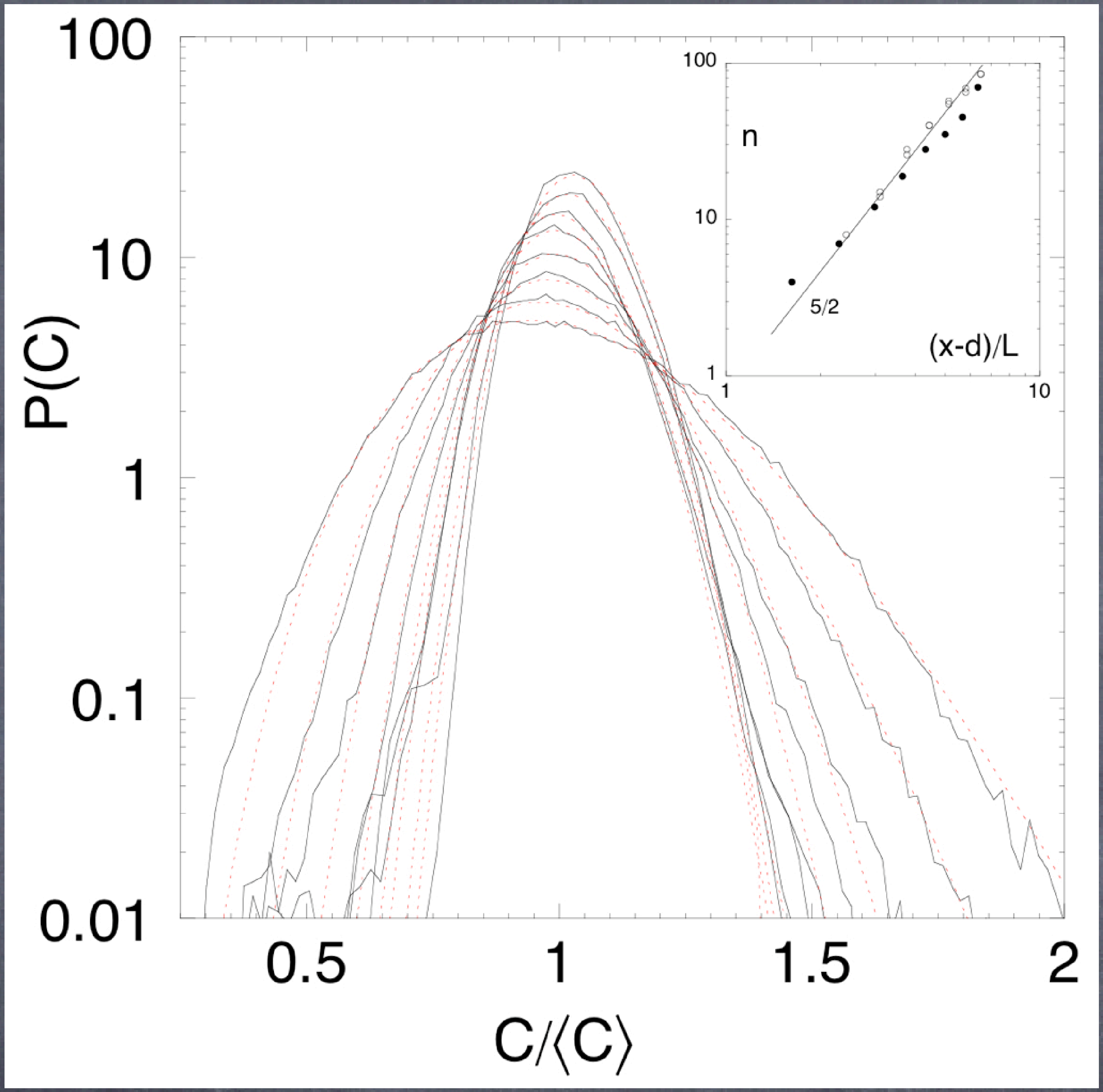
$$\tilde{p} = \left(1 + \langle c \rangle \frac{s}{n}\right)^{-n}$$

if $\langle c \rangle = c^{st} \Rightarrow \gamma = r \; ; \; \frac{dn}{dt} = \gamma n$

Thus: $p\left(x = \frac{c}{\langle c \rangle}\right) = \frac{n^n}{\Gamma(n)}\, x^{n-1} e^{-nx}$

Gamma Distributions

$$\gamma = \frac{5/2}{t} \longrightarrow n(t) \sim t^{5/2}$$

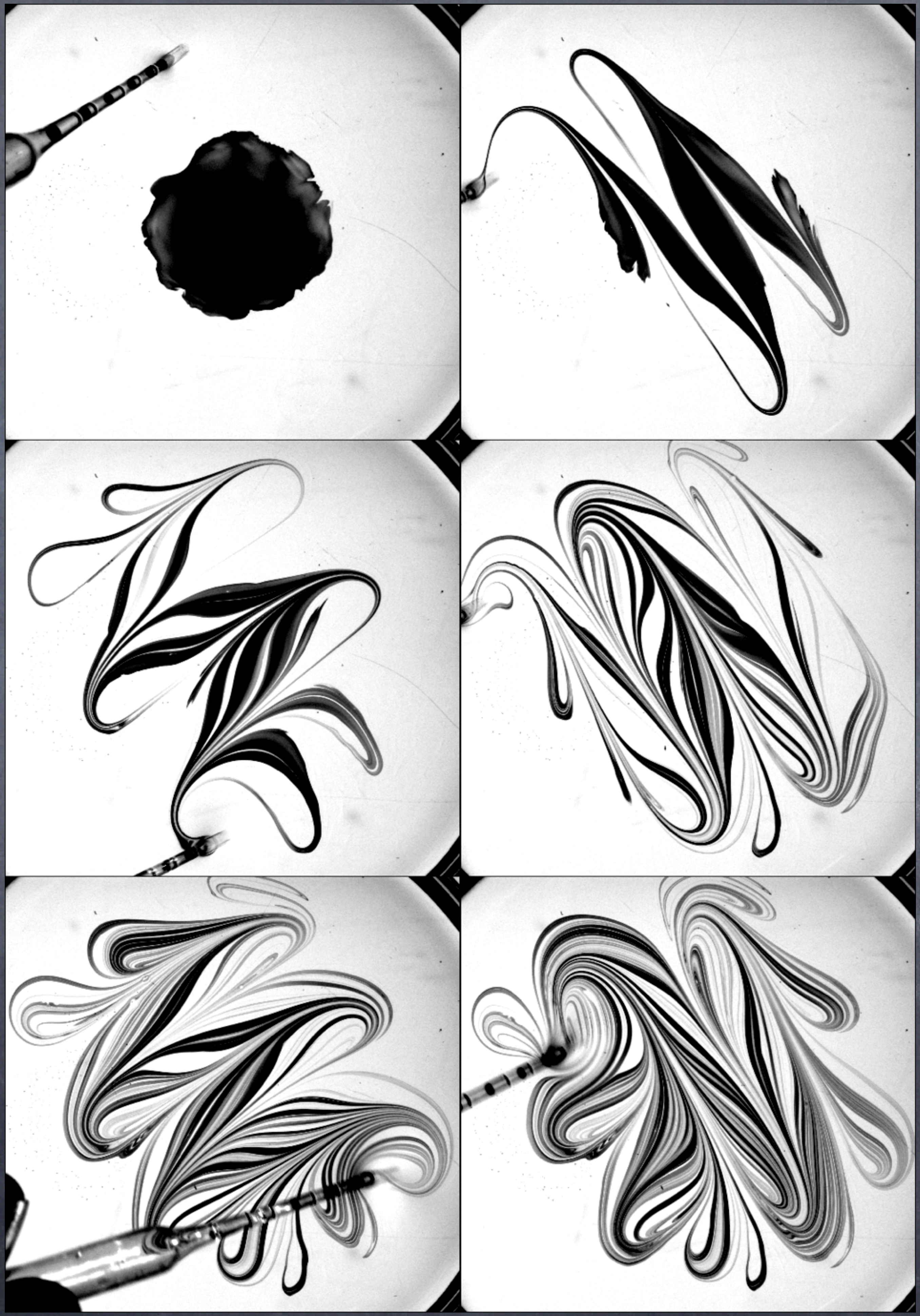


$$n \sim t^{2+1/2}$$

$$P(X = C/\langle C \rangle) = \frac{n^n}{\Gamma(n)} \, X^{n-1} \, e^{-nX}$$

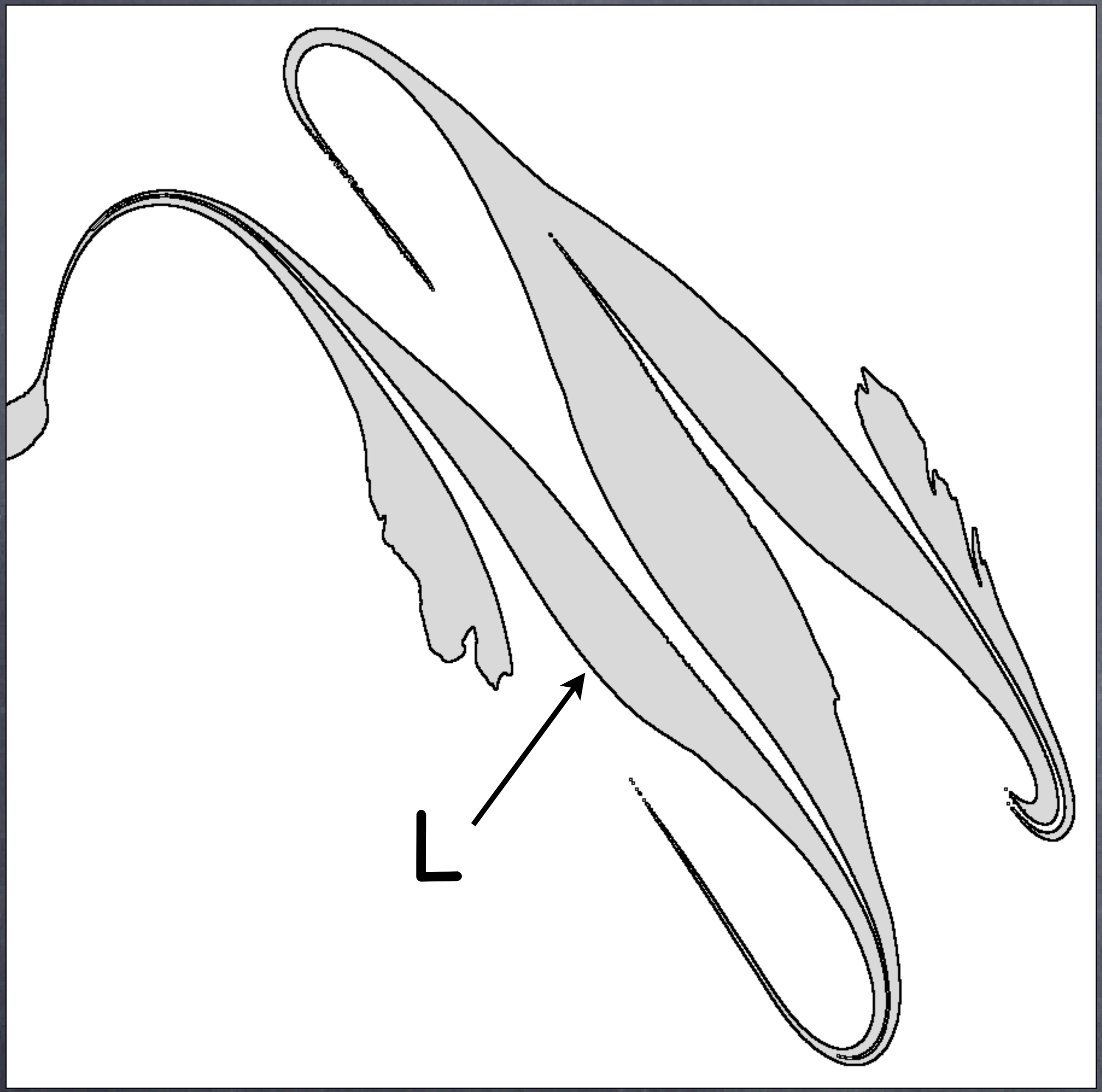
Villermaux & Duplat, PRL 91, 184501 (2003)

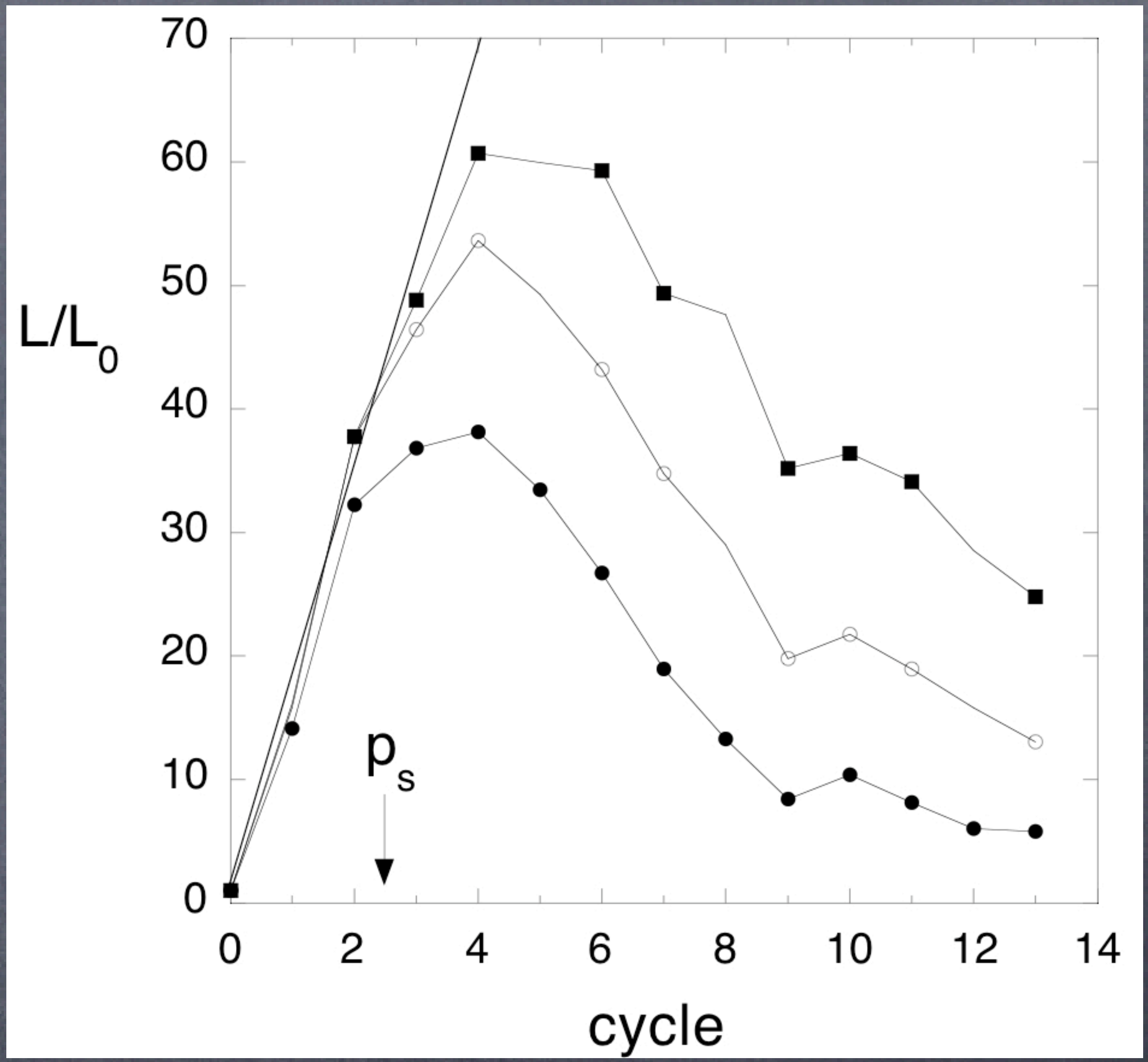
Stirring Protocol



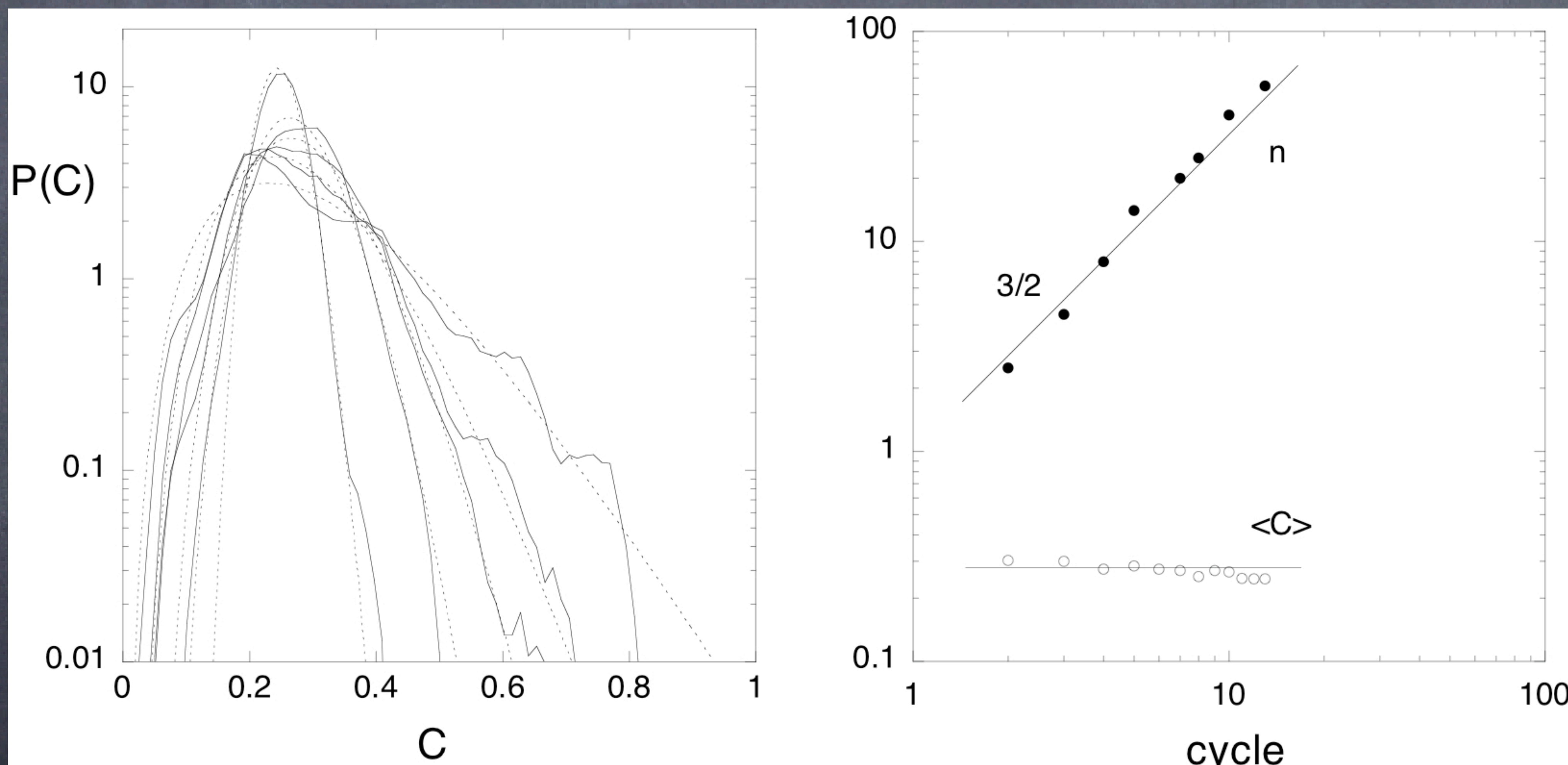
$Re = 0.1$

Material Contours lengths





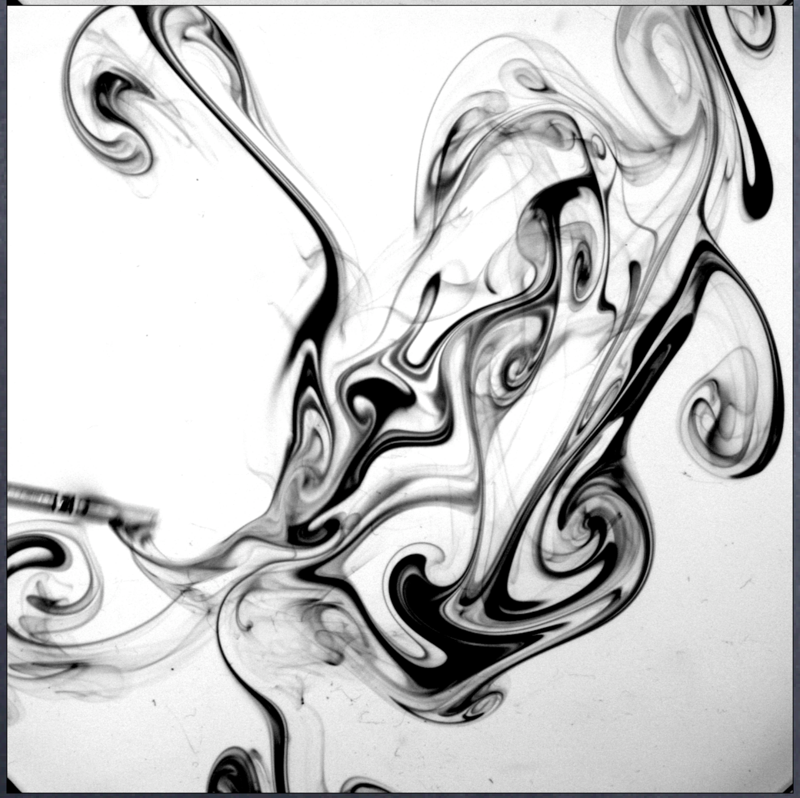
Concentration distribution

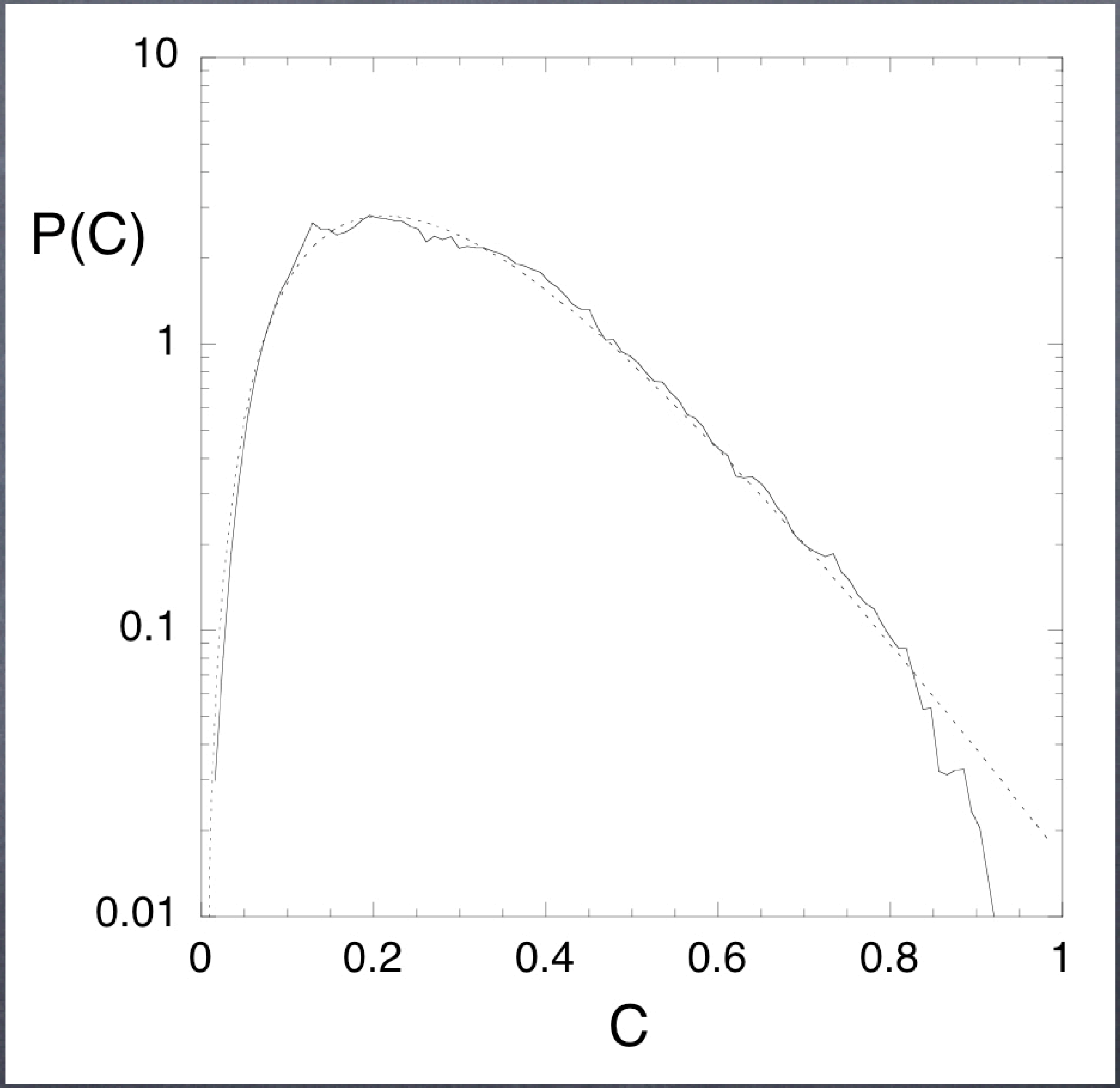


$$n \sim (\text{cycle})^{1+1/2}$$

Somewhat more randomness

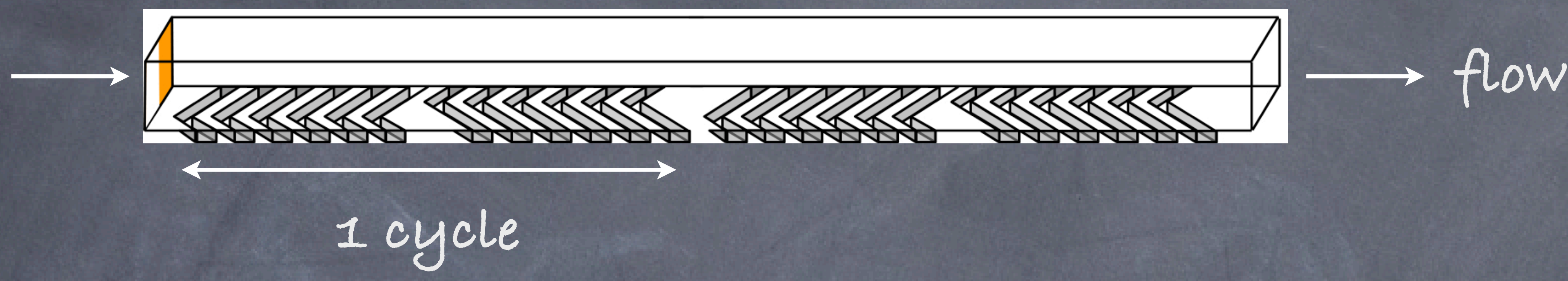
$$Re = 50$$

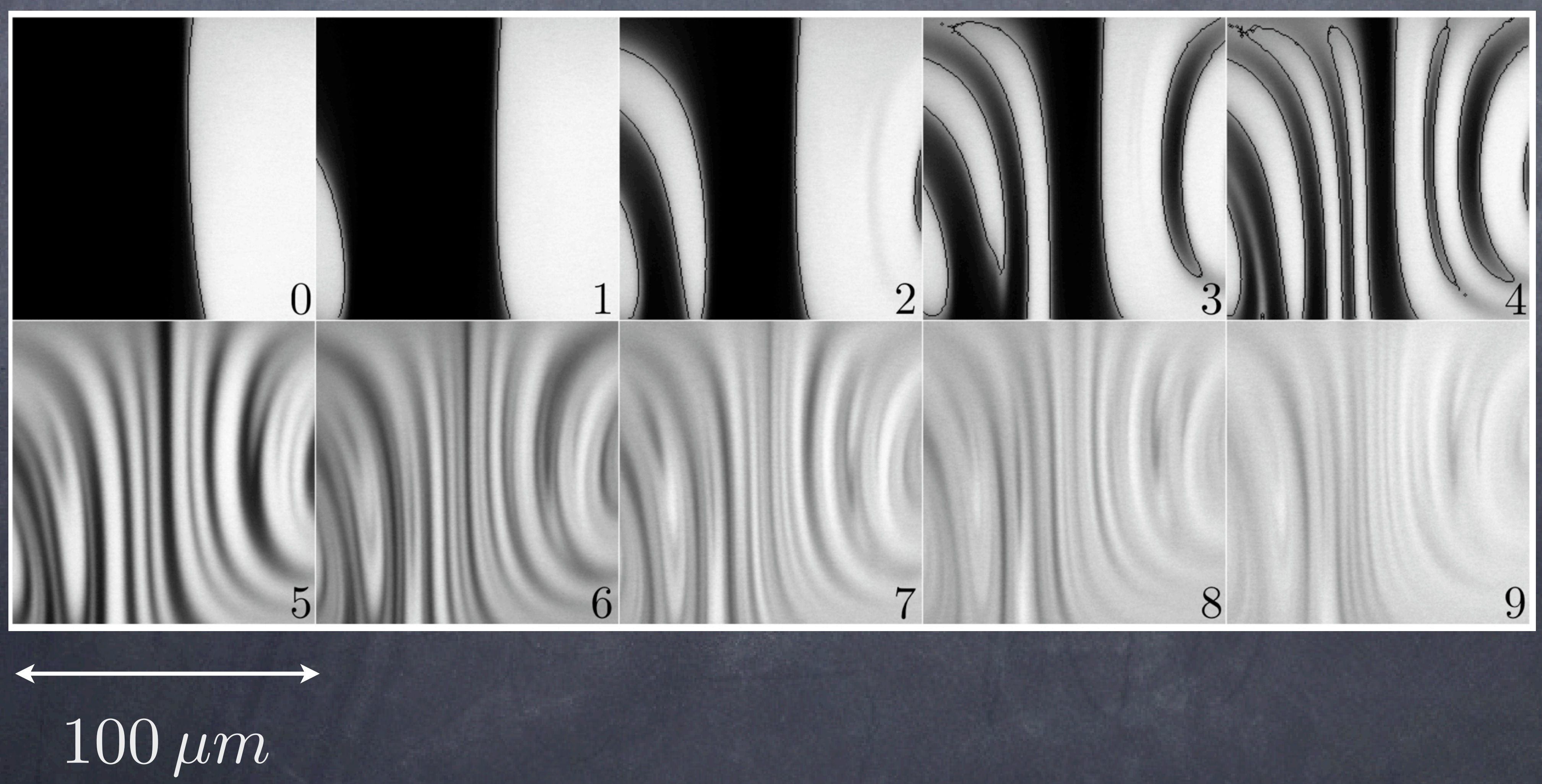




Smoother fit

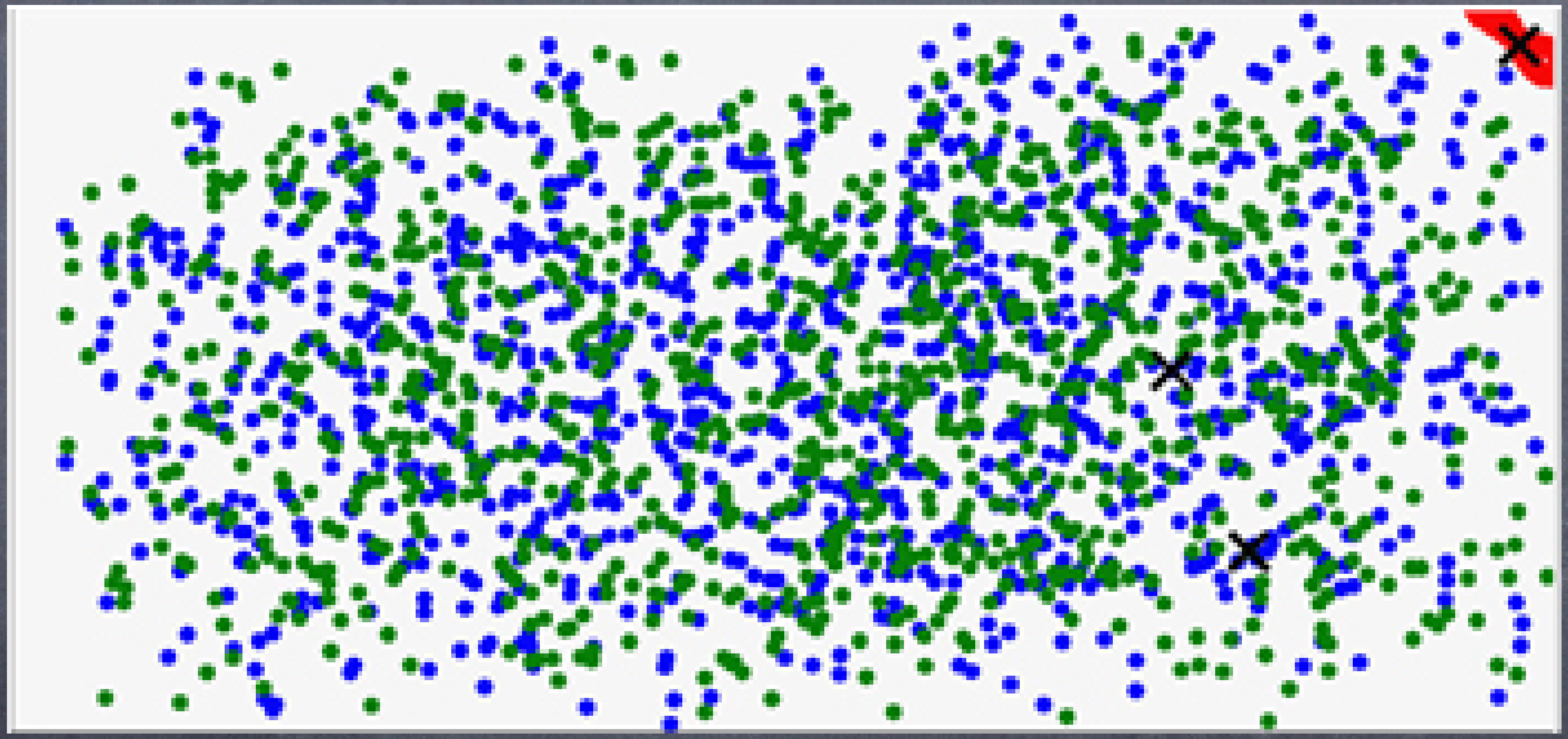
The ‘herringbone’ channel (Stroock & al. Science (2002))





Poincaré section of initially segregated colored puffs

Chaos synonym of random interactions

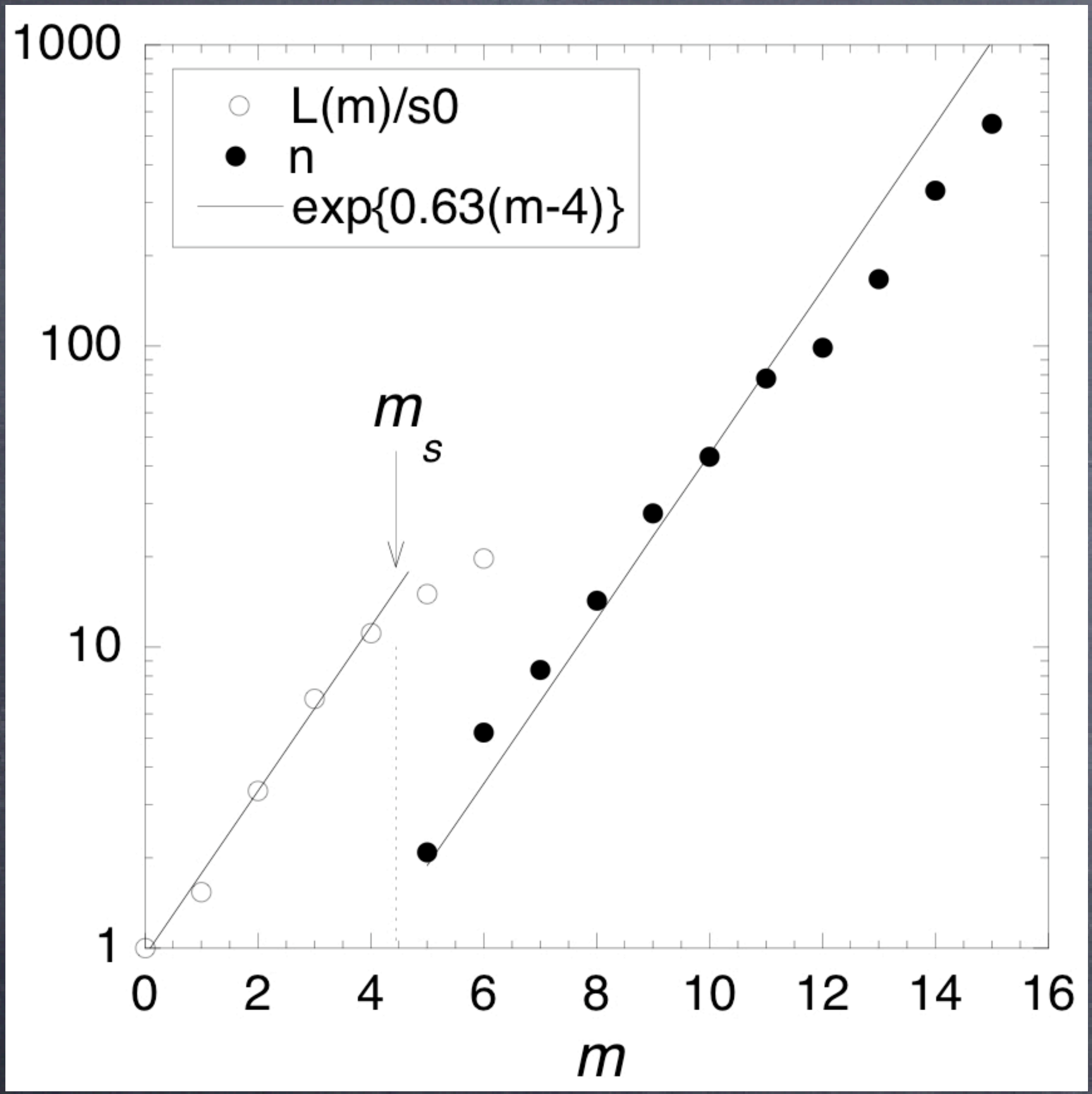


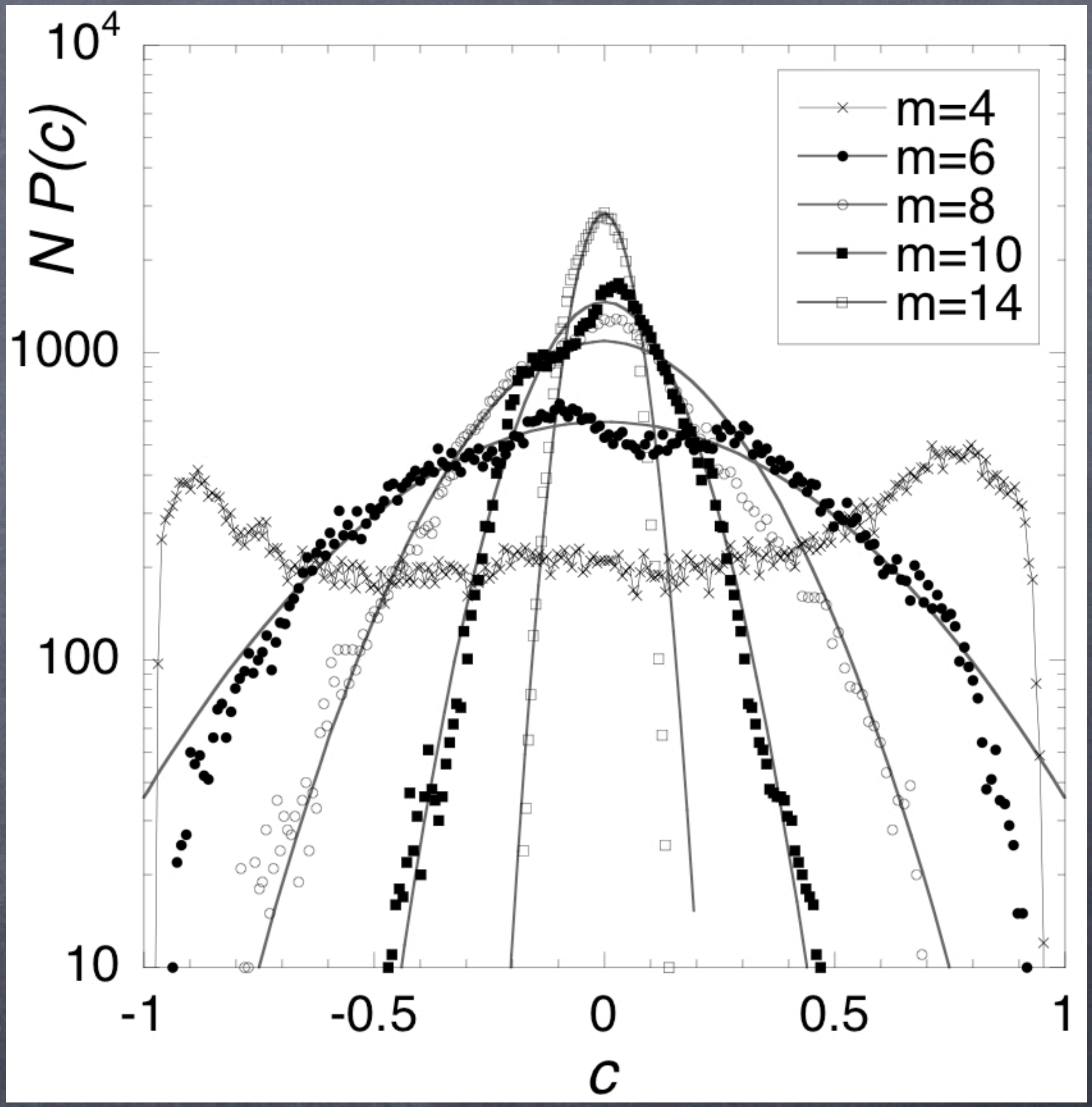
Stroock & McGraw, Proc. Roy. Soc. (2004)

Convolutions about the mean: $c = C - \langle C \rangle$

$\langle C \rangle = 1/2$

P(c)=Bessel functions

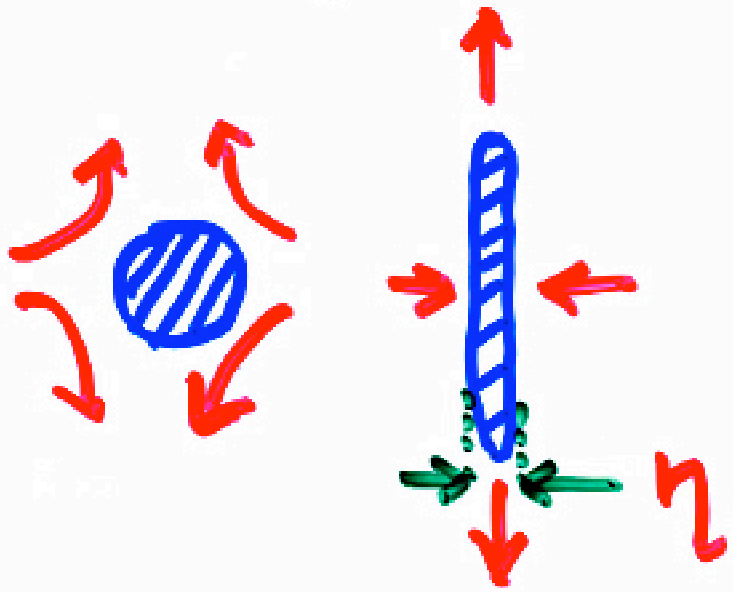




Villermaux & al., PRL (2007)

A Consequence

Equilibrium scales



$$\frac{\eta^2}{D} \sim \frac{r}{\delta u(r)}$$

- if $\delta u(r) \sim \gamma r$

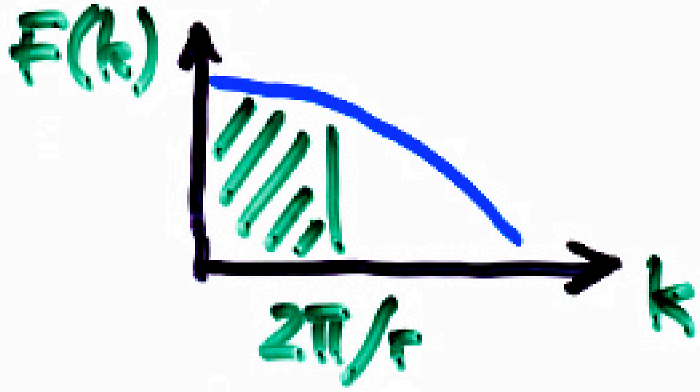
$$\eta \sim \sqrt{\frac{D}{\gamma}}$$ **Batchelor**

- if $\delta u(r) \sim (\varepsilon r)^{1/3}$

$$\eta \sim \left(\frac{D^3}{\varepsilon}\right)^{1/4}$$ **Corrsin-Obukhov**

Coarse grained field

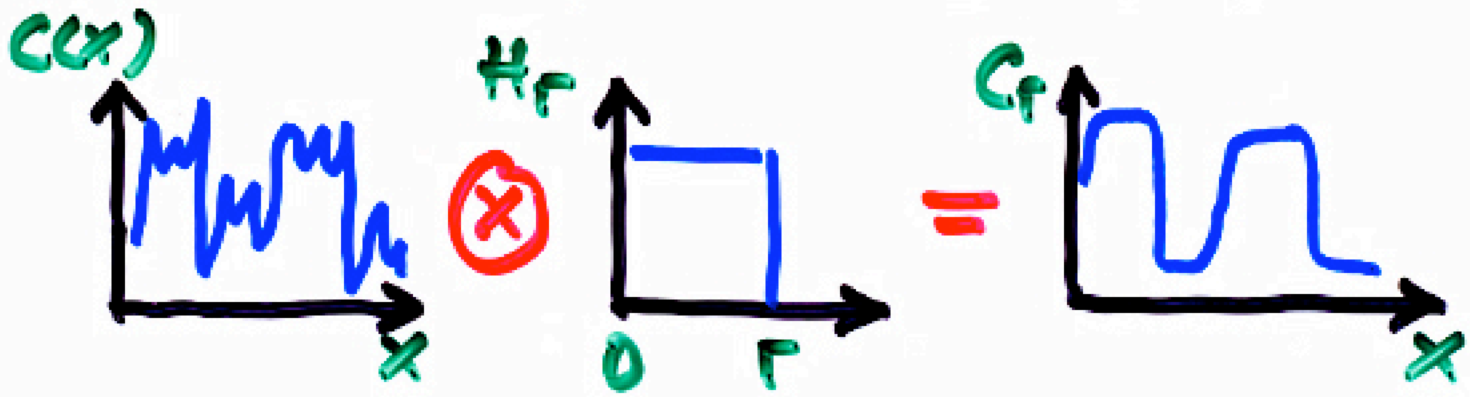
Spectrum: $\left| \int C(r) e^{ikr} dr \right|^2 = F(k)$

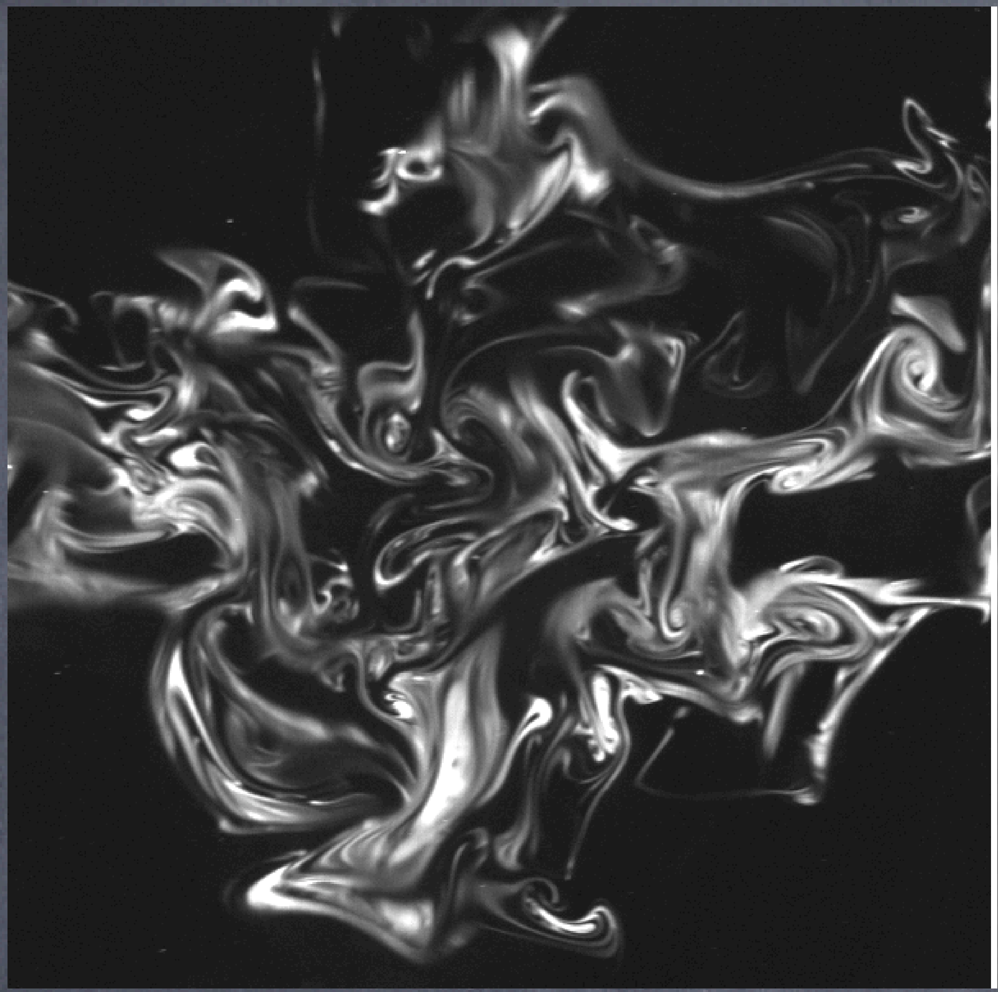


Corsed grained variance :

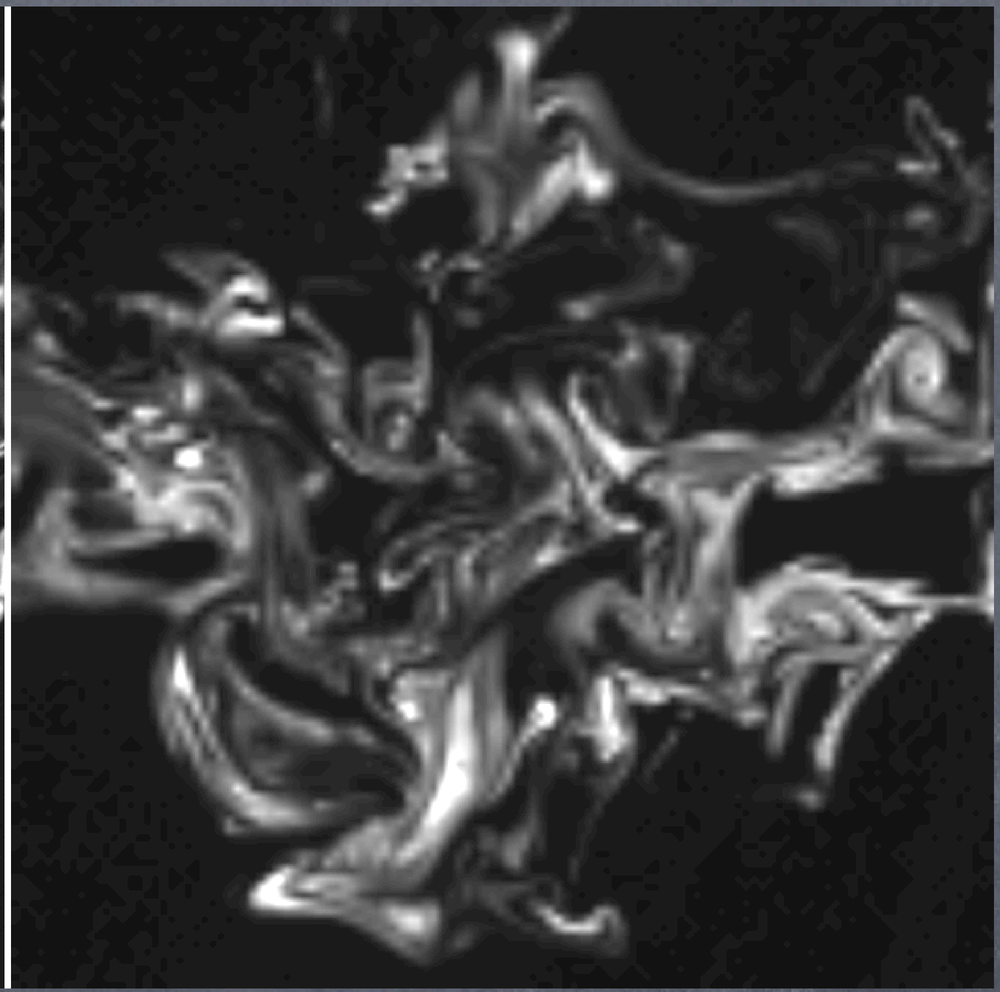
$$V_>(r) = \langle (C_r - \langle C_r \rangle)^2 \rangle$$

where $C_r = C(x) \otimes H_r(x)$



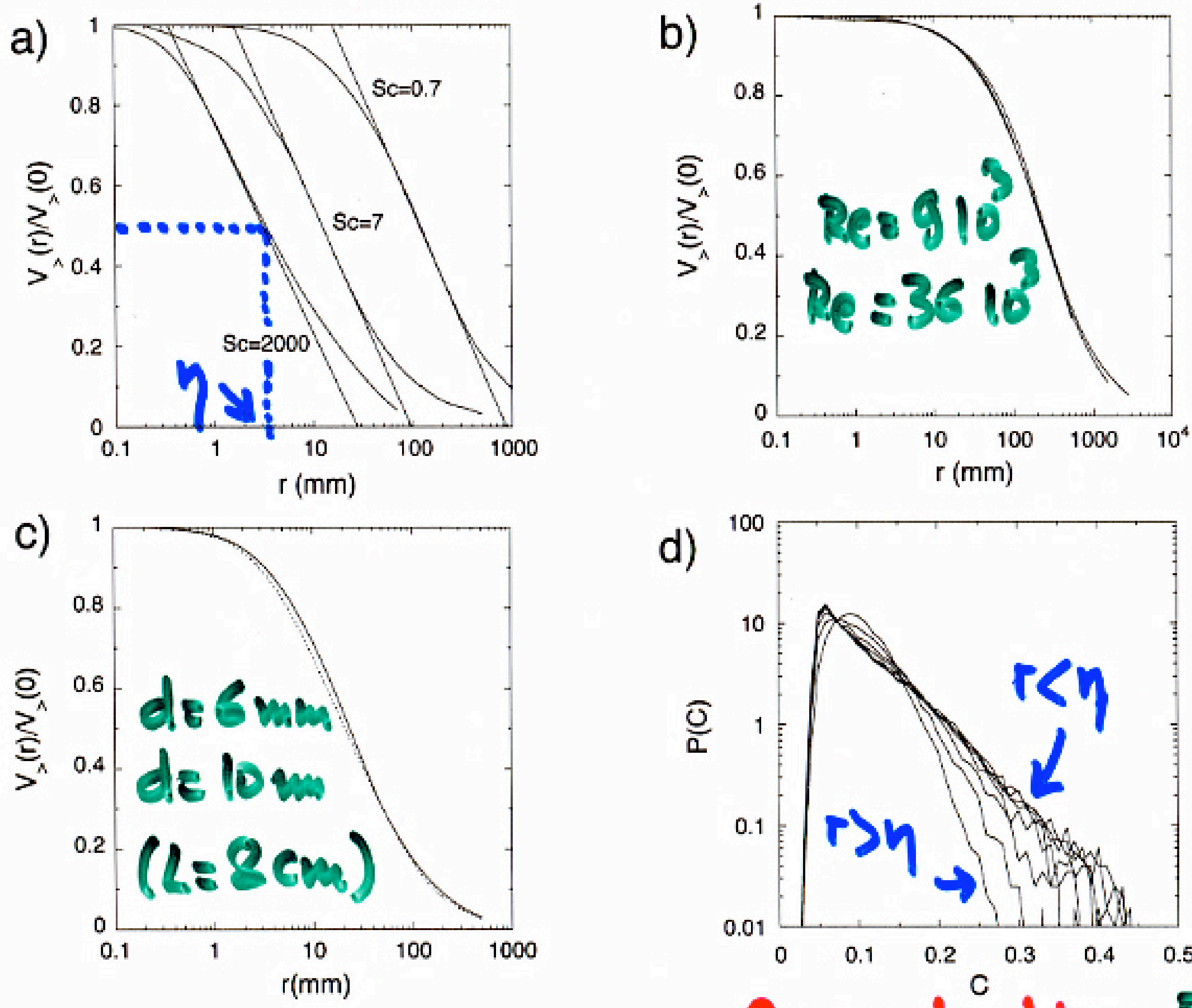


original



coarsened

Variance $V_>(r)$



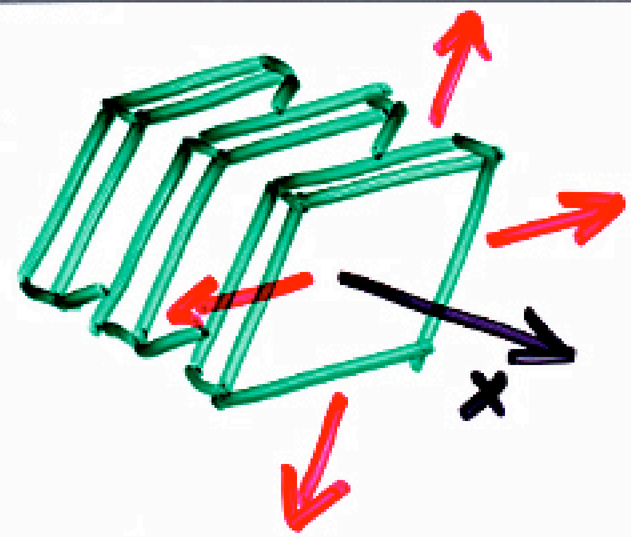
$V_>(r)$ depends on S_c a)

is independent of

Re b)

Source size c)

Concentration Pdf



initially: $C(x,t=0) = 1+\cos(k_0 x)$

then $\partial_t C - \partial_t \ln k(t) \cdot \partial_x C = D \partial_{x^2} C$

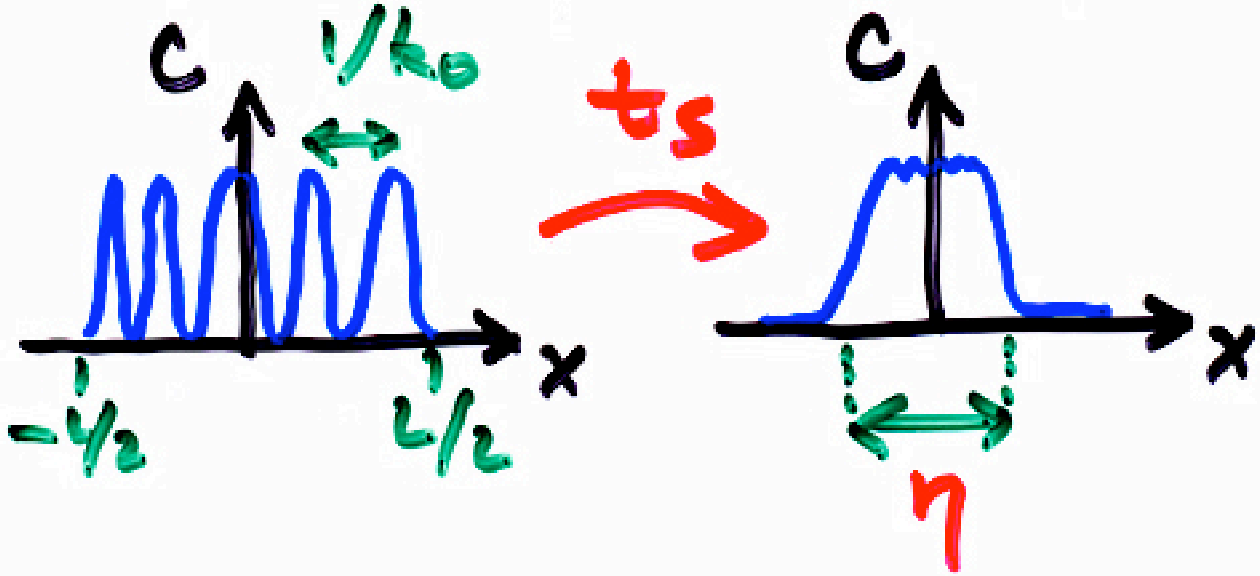
with $k(t) = k_0(1+\sigma t)^2$

$$\tau = \int dt' k(t')^2 = D k_0^2 \left(1 + \frac{2}{3}(\sigma t)^2 + \frac{1}{5}(\sigma t)^4\right)$$

$$\xi = x\, k(t)$$

$$\partial_\tau C = \partial_{\xi^2} C$$

pure diffusion equation



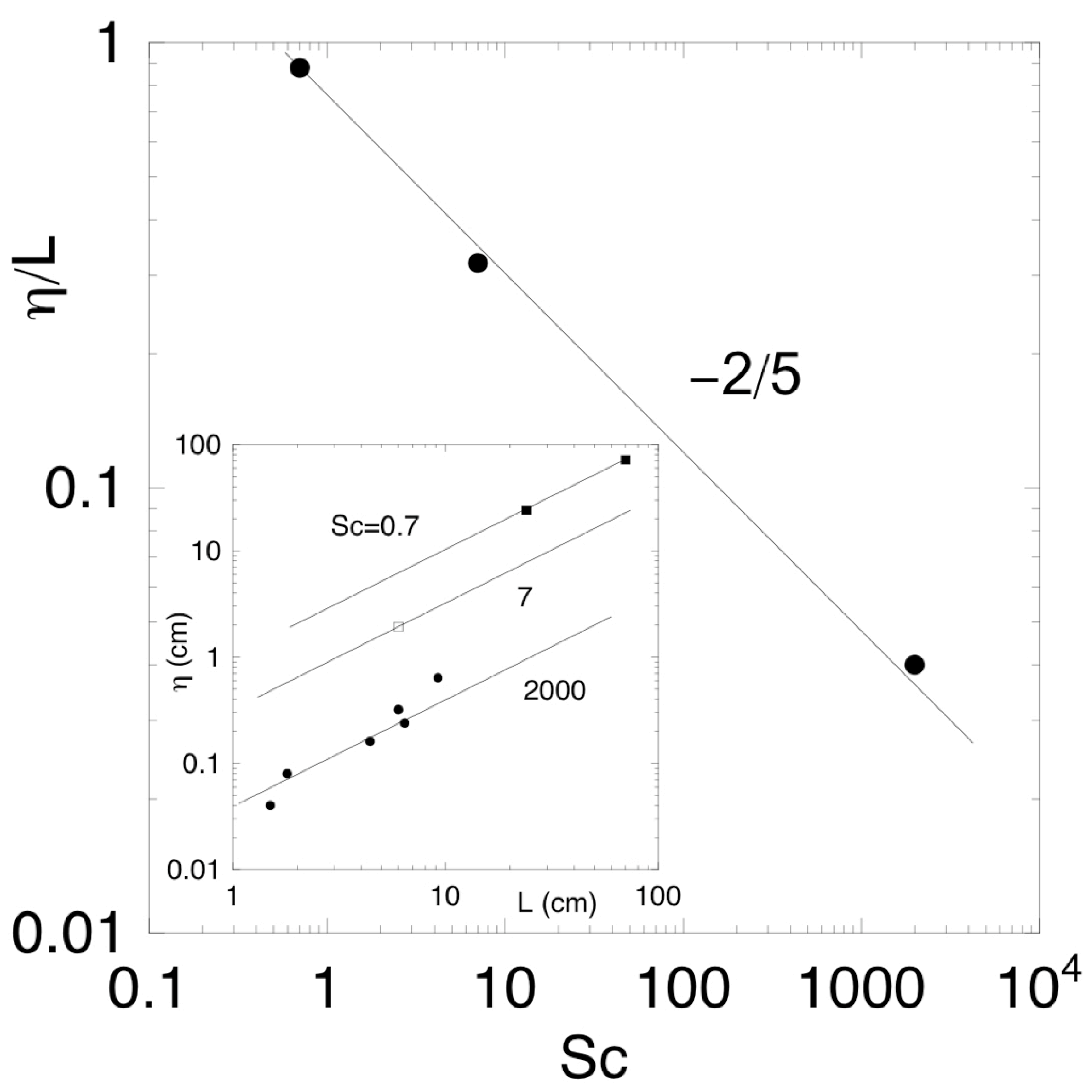
mixing time: $t_s = \frac{1}{\sigma} Pe^{1/5}$; $Pe = \frac{\sigma}{D k_0^2} \gg 1$

Taylor scale : $k_0 = \sqrt{\frac{\sigma}{\nu}}$

$$\eta = \frac{L}{(1+\sigma t_s)^2} \approx L S_c^{-2/5}$$

Coarse grained scale

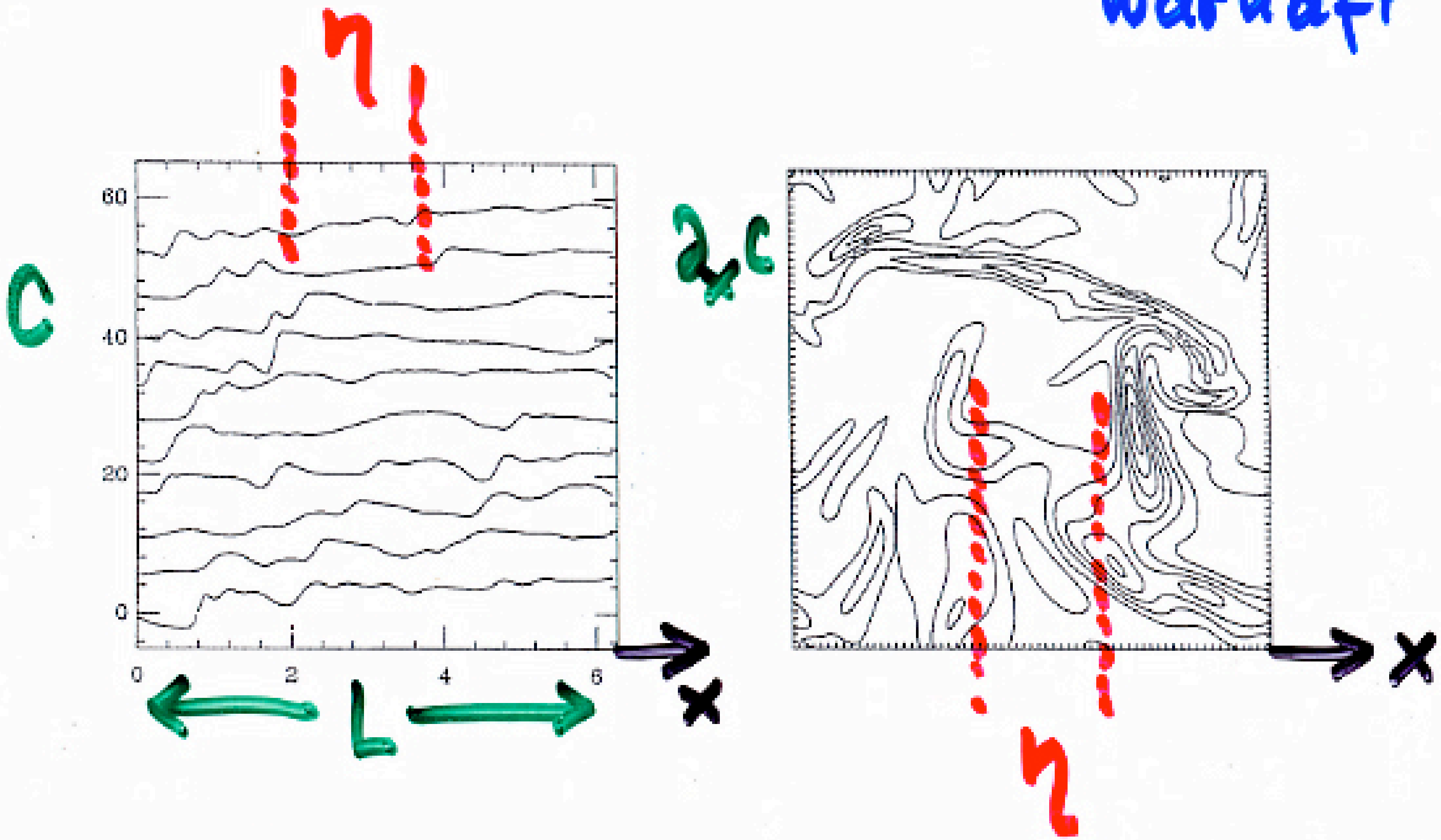
Natural coarse grained scale $\eta = L\,Sc^{-2/5}$



Villermaux & Duplat PRL 97, 144506 (2006) 'plateaux' in shear flows

Pumir '94

Warhaft '00



Mixing is not only an academic topic:
Search for the source !

('Infotaxis', Vergassola & al., Nature 2007)