Power laws and motion estimation from images

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Collaboration with "Universidad de Buenos Aires" (Pablo Mininni)

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  - direct physical-based observation model, *prior* regularity model on motion
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▶ Power law priors in motion estimation

Motion estimation with power law priors

Motion $\mathbf{v}$ estimation by minimization of a global energy on the image :

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but depends on weight α and disconnected from physics !

Probability Distribution Function (PDF) of velocity increments :

- velocity increments :

$$\delta v_{||}(\ell, \mathbf{s}, \mathbf{n}) = (\mathbf{v}(\mathbf{s} + \ell\mathbf{n}) - \mathbf{v}(\mathbf{s})) \cdot \mathbf{n}$$

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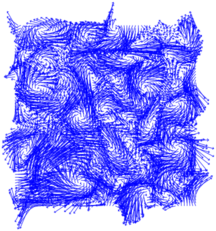
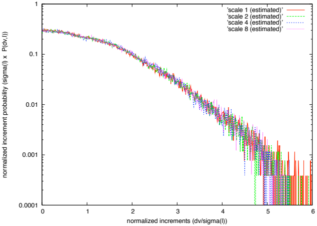
- self-similarity of the PDF (homogeneous and isotropic) at scale ℓ : $p_\ell(\delta v_{||}(\ell))$





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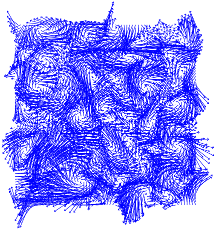

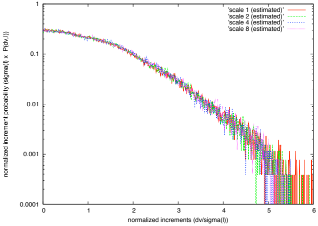

- Third order structure function (3-rd order moment) follows a power law :

$$\begin{aligned} \mathbb{E}[\delta v_{||}(\ell)^3] &= \int_{\mathbb{R}} \delta v_{||}(\ell)^3 p_\ell(\delta v_{||}(\ell)) d\delta v_{||}(\ell) \\ &\propto \beta_3 \ell^{\zeta_3} \end{aligned}$$

where β_3 et ζ_3 are the scaling law parameters

In particular,

- K41law of *Kolmogorov41* for 3D Navier-Stokes :

$$\mathbb{E}[\delta v_{||}(\ell)^3] = -\frac{4}{5}\epsilon\ell, \text{ \textit{in the inertial zone}}$$

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- Model of *Lindborg01* for atmospheric flows :

$$\mathbb{E}[\delta v_{||}(\ell)^3] = -\epsilon\ell + \frac{1}{8}\epsilon_\omega \ell^3,$$

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Strict self-similarity :

$$\beta\ell^\zeta \sim \mathrm{E}[\delta v_{||}(\ell)^2] = \mathrm{E}[\delta v_{||}(\ell)^3]^{\frac{2}{3}} \sim \beta\ell^{\frac{2\zeta_3}{3}}$$

- but intermittency ⇒ non-strict self-similarity

Self-similar constraint at scale ℓ :

- 2-nd order moment : mean over the image support Ω and over directions $\mathbf{n}$ (horizontal, vertical and diagonal) :

$$\mathrm{E}[\delta v_{||}(\ell)^2] \simeq \frac{1}{|\mathbf{n}||\Omega|} \int_{\Omega} \int_{\mathbf{n}} \left(\delta v_{||}(\ell, \mathbf{s}, \mathbf{n}) \right)^2 d s d\mathbf{n}$$

- The turbulent velocity field $\mathbf{v}$ must respect the contraint :

$$g_{\ell}(\mathbf{v}, \beta, \zeta) = \frac{1}{2}(\mathrm{E}[\delta v_{||}(\ell)^2] - \beta \ell^{\zeta}) = 0$$

depending on scaling law parameters (β, ζ).

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Problem (P) : minimization of image observation model subject to contraints $\{g_{\ell}(\mathbf{v}, \beta, \zeta)\}$:

$$(P) \begin{cases} \min_{\mathbf{v}} \ f_d(I, \mathbf{v}) \\ s.t. : \\ g_{\ell}(\mathbf{v}, \beta, \zeta) = 0, \quad \forall \ell \in \mathbf{l} \\ \mathbf{v} \in \mathbb{R}^n \end{cases} .$$

where $\mathbf{l}$ is the power law scale range.

- Lagrangian associated to problem (P) :

$$L(\mathbf{v}, \boldsymbol{\lambda}, \beta, \zeta) = f_d(I, \mathbf{v}) + \sum_{\ell \in \mathbf{I}} \lambda_\ell g_\ell(\mathbf{v}, \beta, \zeta), \quad \boldsymbol{\lambda} = \{\lambda_\ell\}.$$

Dual problem (D) : find the "saddle point" $(\mathbf{v}^*, \boldsymbol{\lambda}^*)$ of the lagrangian

$$L(\mathbf{v}^*, \boldsymbol{\lambda}^*, \beta, \zeta) = \max_{\boldsymbol{\lambda}}\{\min_{\mathbf{v}} L(\mathbf{v}, \boldsymbol{\lambda}, \beta, \zeta)\}$$

- Solved with Uzawa algorithm details

Objective : remove the prior dependance by

- selecting the most likely prior power law $\mathcal{M}(\beta, \zeta) = \beta \ell^{\zeta}$ directly from the image !
- Thus, characterize turbulence
  - flow regularity
  - flux across scales

Likelihood of power law priors

Motion estimation for variable hierarchy :

$$\underbrace{I}_{\textit{image}} \longrightarrow \underbrace{\mathbf{v}}_{\textit{motion}} \longrightarrow \underbrace{\gamma}_{\textit{variance inv.}} \longrightarrow \underbrace{\mathcal{M}}_{\textit{scaling prior}}$$

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- **1-st level :** Solution of (D) reached at the posterior max (MAP) of **v** :

$$\underbrace{-\log p(\mathbf{v}|I,\gamma,\mathcal{M})}_{\textit{posterior energy}} \propto L(\mathbf{v},\boldsymbol{\lambda}^*,\mathcal{M})$$

  where the likelihood and the prior Gibbs energies are quadratic :

$$\underbrace{-\log p(I|\mathbf{v},\gamma)}_{\textit{likelihood energy}} \propto \gamma f_d(I,\mathbf{v})$$

$$\underbrace{-\log p(\mathbf{v}|\mathcal{M})}_{\textit{prior energy}} \propto \sum_{\ell} \lambda_\ell^* g_\ell(\mathbf{v},\mathcal{M})$$

  (MAP estimate $\mathbf{v}^*$ obtained using conjugate gradients and *Uzawa*).

Obs. model variance estimation :

$$\underbrace{I}_{\textit{image}} \rightarrow \underbrace{\gamma}_{\textit{variance inv.}} \rightarrow \underbrace{\mathcal{M}}_{\textit{scaling prior}}$$

- **2-nd level :** Evidence maximum (ML) of observation model variance inverse γ :

$$p(I|\gamma, \mathcal{M}) = \int_{\mathbb{R}^n} p(I|\mathbf{v}, \gamma, \mathcal{M}) p(\mathbf{v}|\gamma, \mathcal{M}) \mathrm{d}\mathbf{v}$$

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$$\begin{aligned} p(I|\gamma,\mathcal{M}) &= \int_{\mathbb{R}^n} p(I|\mathbf{v},\gamma,\mathcal{M})p(\mathbf{v}|\gamma,\mathcal{M})\mathrm{d}\mathbf{v} \\ \text{evidence} &= \frac{\text{likelihood} \times \text{prior}}{\text{posterior}} = \frac{p(I|\mathbf{v},\gamma)p(\mathbf{v}|\mathcal{M})}{p(\mathbf{v}|I,\gamma,\mathcal{M})} \end{aligned}$$

For Gibbs PDF, evidence is a normalization constant ratio. Thus :

$$\underbrace{\log p(I|\gamma,\mathcal{M})}_{log\ evidence} \propto - \underbrace{\gamma f_d(I,\mathbf{v}^*)}_{obs.\ model\ misfit} + \underbrace{\frac{1}{2}\log\frac{\det \sum_\ell \lambda_\ell^* A_\ell}{\det(A_0 + \sum_\ell \lambda_\ell^* A_\ell)} + \frac{m}{2}\log\gamma}_{log\ Occam\ factor}$$

(ML estimate $\hat{\gamma}$ obtained analytically by canceling $\partial_\gamma \log p(I|\gamma,\mathcal{M})$).

Scaling estimation :

$$\underbrace{I}_{image} \longrightarrow \underbrace{\mathcal{M}}_{scaling\ prior}$$

- **3-rd level :** Maximum Likelihood (ML) of scaling prior $\mathcal{M}$:

$$p(I|\mathcal{M}) = \int_{\mathbb{R}} p(I|\gamma, \mathcal{M}) p(\gamma) d\gamma$$

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$$p(I|\mathcal{M}) = \int_{\mathbb{R}} p(I|\gamma,\mathcal{M})p(\gamma)d\gamma$$

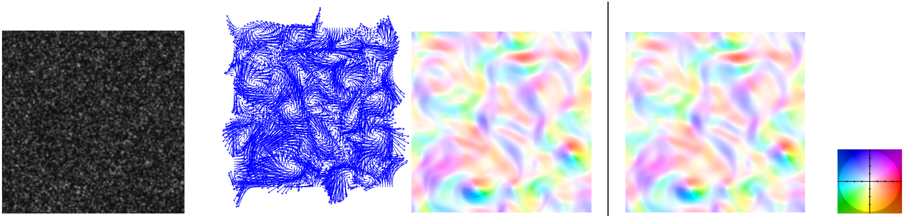
For a flat prior $p(\gamma)$, we obtain the Gaussian approximation :

$$-\log p(I|\mathcal{M}) \propto -\log p(I|\hat{\gamma},\mathcal{M}) - \frac{1}{2}\log(\hat{\gamma} f_d(I,\mathbf{v}^*))$$

(ML optimal scaling prior $\mathcal{M}(\hat{\gamma},\hat{\zeta})$ obtained by sampling the evidence uniformly in the parameter space (γ,ζ)).

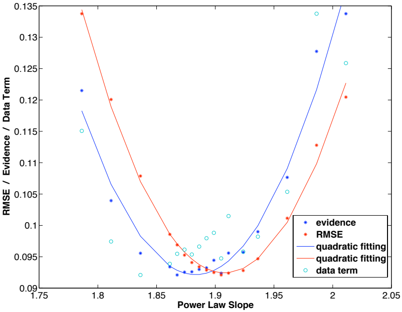
Experimental evaluation

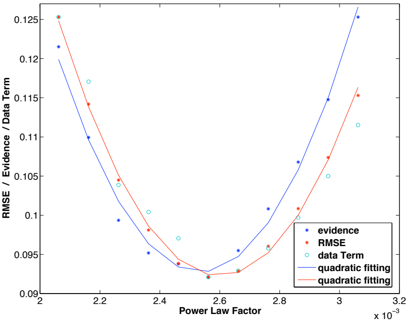
- Forced 2D DNS ($Re = 3000$), dissipative or enstrophy cascade at small scales : $\zeta \sim 2$
- Synthesis of a particule image sequence *[Carlier05]*
- Power law priors in the scale range of [1,10] pixels



Left : *particle image obtained by DNS of 2D Navier-Stokes equations & true velocity field.*

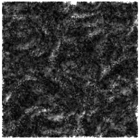
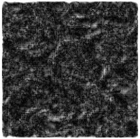
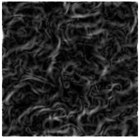
Right : *estimated velocity field.*



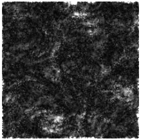
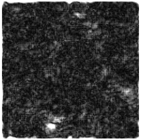
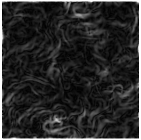


Power law model evidence. *Behavior of data term and minus the logarithm of the evidence w.r.t slope* ζ *(***left***) and factor* β *(***right***) in comparison to the RMSE.*

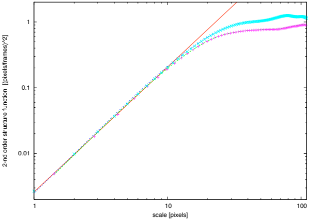
Barron angular error : 4.2656° 4.3581° 3.0485° 2.8836°

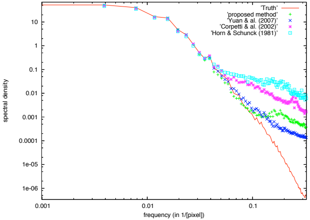

RMSE : 0.138501 0,13402 0.09602 0.09141


Horn & Schunck (1981) (gradient penalization) | Corpetti & al. (2002) (div-curl reg.) | Yuan & al. (2007) (zero div & curl reg.) | proposed method (self-similar reg.)



2-nd order structure function



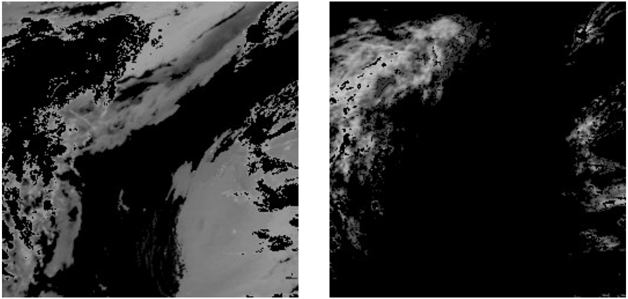
Energy spectrum

- MSG images and direct physical observation model *[Heas&al. 07]*
- Atmospheric turbulence : energy cascades at small scales $[1, 10]$ km *[Lindborg01]* :

$$\mathrm{E}[\delta v(\ell)^2] \sim \beta \ell^{\frac{2}{3}}, \; \textit{avec } \beta = C\epsilon^{\frac{2}{3}} \; \textit{et } \epsilon = \textit{energy flux across scales (or dissipation rate)}$$

- Self-similarity constraints in the scale range $[1, 4]$ pixels

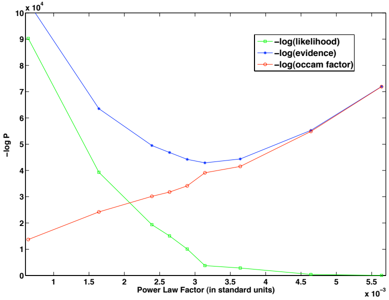
Lower layer	Intermediate layer



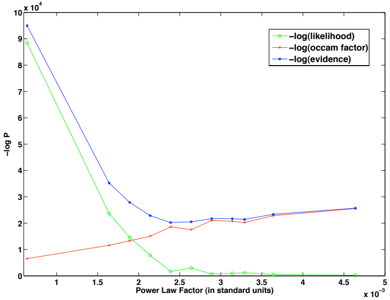
Pressure difference images

Selection by evidence maximization of the flux ϵ in the energy cascade :

Lower layer



Intermediate layer



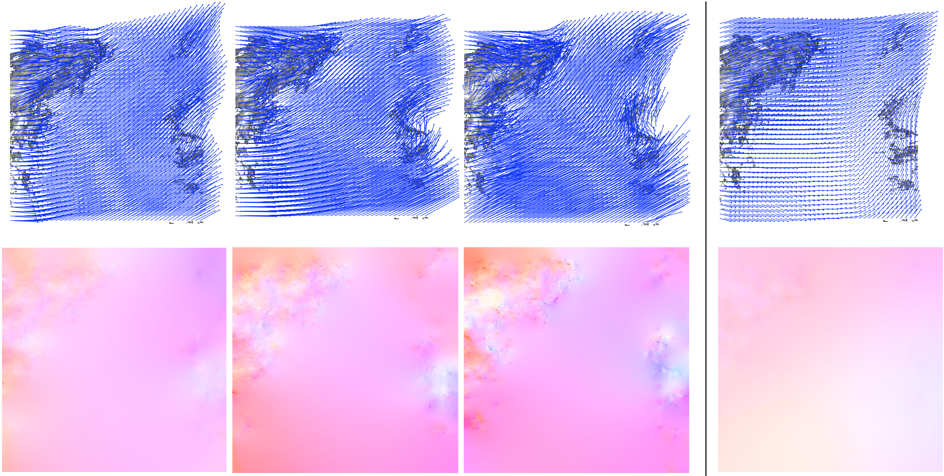
Scaling law model evidence. *Behavior of data term, minus the log evidence and of minus the log of occam factor w.r.t factor* β

- Minimum of evidence $\Rightarrow \hat{\beta}$, yields energy flux across scales estimates :

$$\begin{cases} \hat{\epsilon}^{mid} \simeq 0.79 \times 10^{-5} m^2 s^{-3} \\ \hat{\epsilon}^{low} \simeq 1.20 \times 10^{-5} m^2 s^{-3}. \end{cases}$$

- Same order of magnitude as observed *in situ* by *[Dewan97, Lindborg01]*

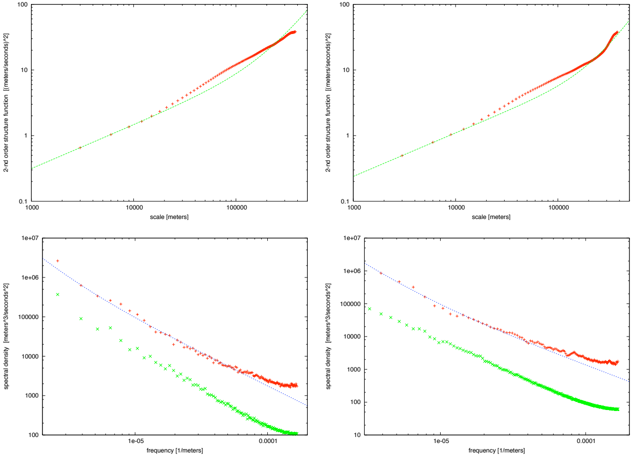
Experimental evaluation - Atmospheric turbulence



$\epsilon = 0.11e-5$ | $\hat{\epsilon} = 0.79e-5$ | $\epsilon = 2.15e-5$ | *[Heas&al. 07]*

Intermediate wind fields for increasing energy flux (left) in comparison to [Heas&al. 07] (right)

Lower layer	Intermediate layer



Second order structure functions & spectra

- Quadratic penalization of motion n-order derivative

For a wavelet ψ with n vanishing moments :

$$\lim_{\ell \to 0} \frac{\mathcal{W}_\ell \mathbf{v}(u,\ell)}{\ell^{n+1/2}} = K\partial^n \mathbf{v}(u)$$

Therefore, a n-order derivative penalization $(\partial^n \mathbf{v}(u))^2$ can be written as :

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- Impose a $\mathbb{C}^n$ regularity by penalizing coarser scales coef.
- Impose self-similarty with quadratic functionals
  - Hard-constraints for p=1 or 2 th order wavelet structure function

$$g_\ell(\mathbf{v}) = \left(\int_{\Omega_\ell} \int_\theta (\mathcal{W}_\ell \mathbf{v})^p d\mathbf{s} d\theta \right) - \beta_p \ell^{\zeta_p}$$

  - Quadratic constraints on 3-rd order wavelet structure function (Gaussian assumption)

$$\mathbb{E}[(\mathcal{W}_\ell \mathbf{v})^3] = 3\mathbb{E}[(\mathcal{W}_\ell \mathbf{v})^2]\beta_1 \ell^{\zeta_1} - 3\mathbb{E}[(\mathcal{W}_\ell \mathbf{v})](\beta_1 \ell^{\zeta_1})^2 + (\beta_1 \ell^{\zeta_1})^3$$

Lagrangian minimisation w.r.t. $\mathbf{v}$:

- Cancelling the functional (quadratic form) gradient

$$\nabla_{\mathbf{v}} L(\mathbf{v}, \boldsymbol{\lambda}) = \nabla_{\mathbf{v}} f_d(I, \mathbf{v}) + \sum_{\ell} \lambda_{\ell} \nabla_{\mathbf{v}} g_{\ell}(\mathbf{v}) = 0.$$

or resolution of a large linear system

$$(A_0 + \sum_{\ell} \lambda_{\ell} A_{\ell})\mathbf{v} = \mathbf{b}_0 + \sum_{\ell} \lambda_{\ell} \mathbf{b}_{\ell},$$

▶ solution $\mathbf{v}^*$ obtained by the Conjugate Gradient Squared (CGS) algorithm.

back

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Maximisation of $\boldsymbol{\lambda}$:

Concavity w.r.t. λ of the dual function

$$w(\lambda) = L(\mathbf{v}^*, \lambda)$$

▶ solution λ^* obtained by a classical gradient algorithm.

back

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or resolution of a large linear system

$$(A_0 + \sum_{\ell} \lambda_{\ell} A_{\ell})\mathbf{v} = \mathbf{b}_0 + \sum_{\ell} \lambda_{\ell} \mathbf{b}_{\ell},$$

▶ solution $\mathbf{v}^*$ obtained by the Conjugate Gradient Squared (CGS) algorithm.

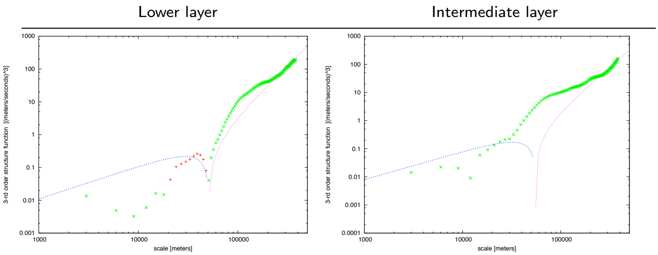
Maximisation of $\boldsymbol{\lambda}$:

Concavity w.r.t. λ of the dual function

$$w(\lambda) = L(\mathbf{v}^*, \lambda)$$

▶ solution λ^* obtained by a classical gradient algorithm.

$\Rightarrow$ **velocity $\mathbf{v}^*$ and optimal weights λ^* of multi-scale regularization.** back



Third order structure functions

Least square estimationof the flux ϵ_ω in the enstrophy cascade :

$$\mathrm{E}[\delta v(\ell)^3] \quad = \quad -\epsilon\ell + \frac{1}{8}\epsilon_\omega \ell^3$$

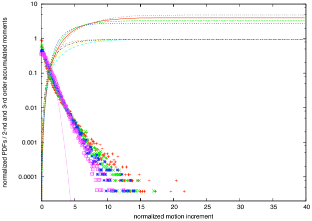
- Least squares using the 3-rd structure functions yields :

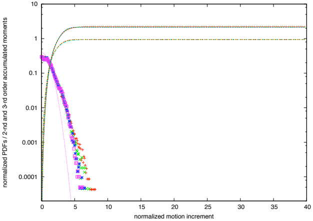
$$\begin{cases} \hat{\epsilon}_\omega^{mid} \simeq 2.58 \pm 0.78 \times 10^{-15} s^{-3} \\ \hat{\epsilon}_\omega^{low} \simeq 4.16 \pm 0.23 \times 10^{-15} s^{-3} \end{cases}$$

- Same order of magnitude as observed by *[Charney71, Lindborg01,Tung03]*
- Direct energy cascade observed only on one layer ...

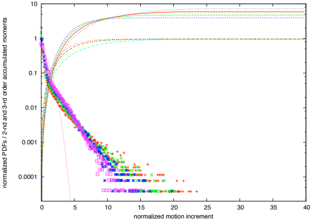
Sufficiently converged statistics ?

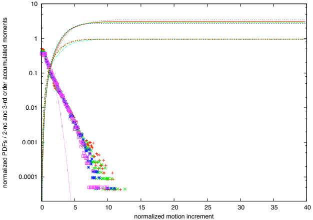
Lower layer : Small scales | Large scales





Intermediate layer : Small scales | Large scales





- Convergence of accumulated moments of second and third order $C_2(z,\ell)$, $C_3(z,\ell)$:

$$C_p(z,\ell) = \int_0^z \left| \frac{\delta v_{||}(\ell)}{\sigma_\ell} \right|^p \sigma_\ell p_\ell \left(\frac{\delta v_{||}(\ell)}{\sigma_\ell} \right) d \left(\frac{\delta v_{||}(\ell)}{\sigma_\ell} \right)$$