Random change of flow topology in 2D and geophysical turbulence

Out of equilibrium statistical mechanics of the large scales of 2D turbulence

F. BOUCHET – Institut Non Linéaire de Nice

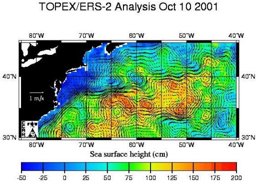
GDR Turbulence, Lyon, avril 2008

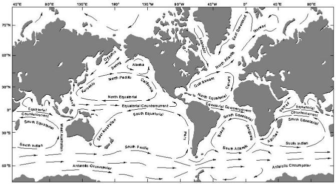
Collaborations

- Statistical mechanics of two dimensional and geophysical flows : F. Gallaire (Lab. Dieudonné Nice), H. Morita (Post-Doc - INLN), F. Rousset (Lab. Dieudonné Nice), E. Simonnet (INLN-Nice), A. Venaille (PHD-codirected by J. Sommeria-Grenoble), (ANR-JC Statflow)
- Statistical mechanics of systems with long range interactions : J. Barré (Lab. Dieudonné Nice), T. Dauxois (ENS-Lyon), S. Ruffo (Florence), D. Mukamel (Weizmann-Israel), Y. Yamaguchi (Kyoto), P.H. Chavanis
- Self gravitating systems (Wasserstein distance and kinetic theory) : Y. Sota

Outline

The Physical Phenomena





Theoretical ideas :

- Self organisation processes. Large number of degrees of freedom (turbulence).
- This has to be explained using statistical physics !!! Equilibrium and Out of equilibrium statistical mechanics.

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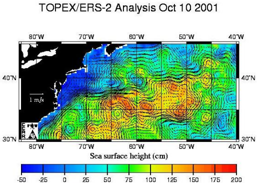

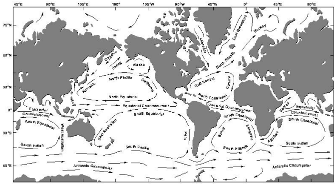

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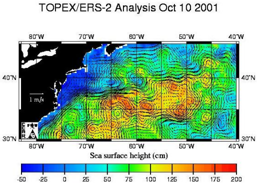

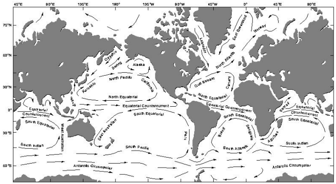

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Out of Equilibrium Phase Transitions in Real Flows

2D MHD experiments (2D Navier Stokes dynamics)

Two-dimensional inverse energy cascade in a square box 141

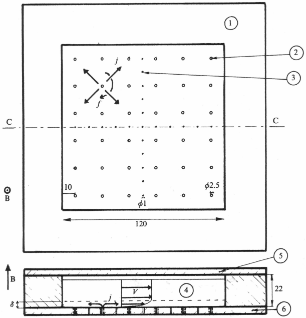


FIGURE 1. The apparatus; the current distribution near one electrode and the velocity profile are schematized. The Hartmann-layer depth is denoted by δ. (1) Copper frame. (2) Electrodes for current injection and electric potential measurements. (3) Electrodes for electric potential measurements only. (4) Mercury. (5) Glass cover. (6) Electrically insulating bottom plate in which electrodes are embedded.

Two-dimensional inverse energy cascade in a square box 15

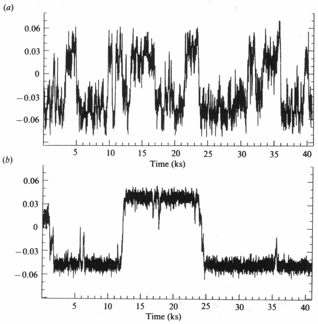
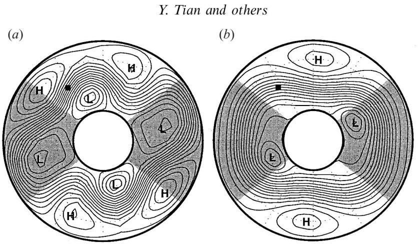


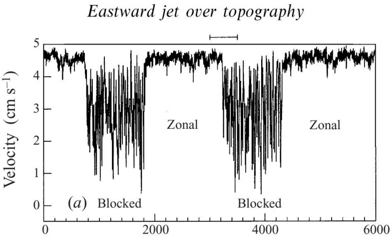
FIGURE 13. Typical time records of the non-dimensional central stream function (free surface, $I = 16$ A). (a) $Rh = 36.8$; (b) $Rh = 39.5$. Notice the very long timescales.

J. Sommeria, J. Fluid. Mech. (1986)

Out of Equilibrium Phase Transitions in Real Flows
Rotating tank experiments (Quasi Geostrophic dynamics)

Transitions between blocked and zonal states

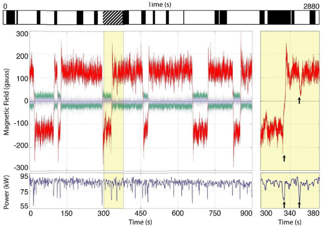




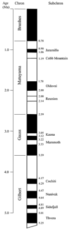
Y. Tian and col, J. Fluid. Mech. (2001) (groups of H. Swinney and M. Ghil)

Random Transitions in Other Turbulence Problems

Magnetic Field Reversal (Turbulent Dynamo, MHD Dynamics)



VKS experiment



Earth

(VKS experiment)

Other examples :

- turbulent convection,
- random changes of paths for the Kurushio current, weather regimes, and so on.

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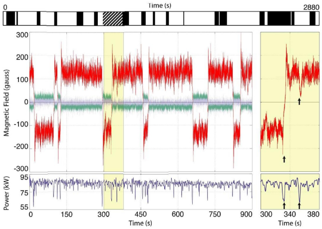

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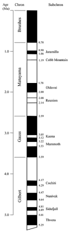

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The 2D Stochastic-Navier-Stokes (SNS) Equations

- The simplest model for two dimensional turbulence
- Navier Stokes equation with a random force

$$\frac{\partial \omega}{\partial t} + \mathbf{u}.\nabla \omega = \nu \Delta \omega - \alpha \omega + f_s \quad (1)$$

where f_s is a random force. α is the Rayleigh friction coefficient.

- An academic model with experimental realizations (Sommeria and Tabeling experiments, rotating tanks, magnetic flows, and so on).

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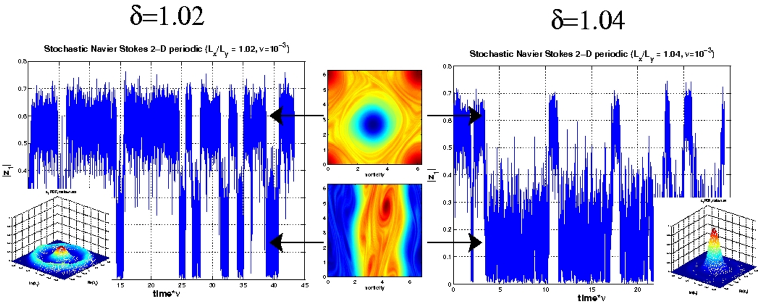
Out of Equilibrium Phase Transitions in the 2D SNS Eq.

Random change of flow topology



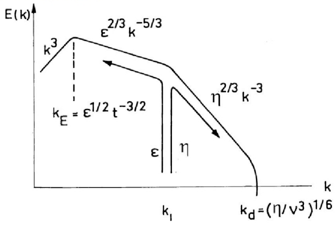
Order parameter : $z_1 = \int dxdy \, \exp(iy)\omega(x,y)$.

For unidirectional flows $|z_1| \simeq 0$, for dipoles $|z_1| \simeq 0.6 - 0.7$.

Outline

Inverse Energy Cascade

Self similar statistics of inertial scales of 2D flows



Sketch of the double cascade in 2D turbulence

- Self similar statistics - Energy spectrum and velocity increments.

Universal Inverse Energy Cascade or Large Scale Energy Condensation

- In order to observe this self-similar regime, Rayleigh friction has to be sufficiently large, in order to prevent the formation of coherent structure at large scales.
- This regime is nearly never realized in real or experimental flows. It clearly does not describe large scale statistics.
- In real flows, one observes energy condensation at largest scales
- We still have an inverse energy cascade but now driven by large scale (no more universal).

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Robert-Sommeria-Miller (RSM) Theory

Equilibrium statistical mechanics : the most probable vorticity field

- A probabilistic description of the vorticity field ω : $\rho(\mathbf{x},\sigma)$ is the local probability to have $\omega(\mathbf{x})=\sigma$ at point $\mathbf{x}$
- A measure of the number of microscopic field ω corresponding to a probability ρ :

Maxwell-Boltzmann Entropy: $\mathcal{S}[\rho] \equiv -\int_{\mathcal{D}} d\mathbf{x} \int_{-\infty}^{+\infty} d\sigma\, \rho \log \rho$

- The microcanonical RSM variational problem (MVP) :

$$S(E_0,d) = \sup_{\{\rho|N[\rho]=1\}} \left\{ \mathcal{S}[\rho] \mid E[\overline{\omega}] = E_0\,, D[\rho] = d \right\} \text{ (MVP).}$$

- Critical points are stationary flows of Euler's equations :

$$\omega = f(\psi)$$

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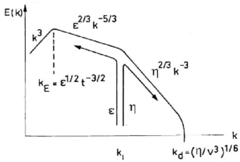
What are Real Flow Regimes

Inverse energy cascade or equilibrium statistical mechanics ?



Inverse cascade

Equilibre statistique

REAL FLOWS ?

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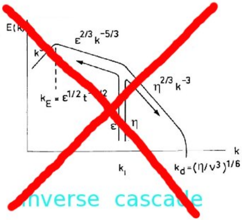

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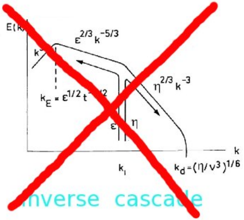


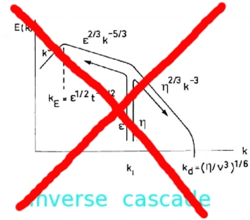
Equilibre statistique

REAL FLOWS ?

Equilibrium statistical predicts stationary flows. It does not take into account forces and dissipation.

What are Real Flow Regimes

Inverse energy cascade or equilibrium statistical mechanics ?



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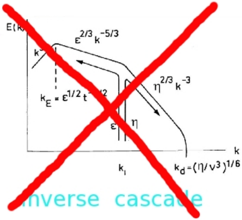
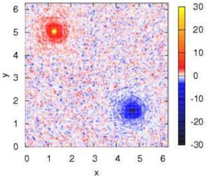


REAL FLOWS ?

Real flows : out of equilibrium statistical mechanics or inverse cascade governed by large scales (and not self similar).

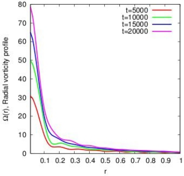
Numerical Simulation of the 2D Stochastic-NS Eq.

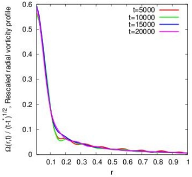


Self similar growth of a dipole structure, for the 2D S-NS equation

Left : vorticity field

Bottom : vorticity profiles





M Chertkov, C Connaughton, I Kolokolov, V Lebedev (nlin.CD/0612052, PRL 2007) (Los Alamos)

Outline

The 2D Stochastic Navier-Stokes Equation

- The 2D Stochastic Navier Stokes equation :

$$\frac{\partial \omega}{\partial t} + \mathbf{u}.\nabla\omega = \nu\Delta\omega - \alpha\omega + \sqrt{2\alpha}f_s \qquad (2)$$

where f_s is a random force (white in time, smooth in space).

- The scaling of the forcing is such that the average energy (enstrophy) is of order one (this is not arbitrary, just a change of time unit).
- We use very small Rayleigh friction, to observe large scale energy condensation
- We study the limit : $\lim_{\alpha\to 0}\lim_{\nu\to 0}$ $(\nu \ll \alpha)$ $(Re \gg R_\alpha \gg 1)$ (Weak forces and dissipation).
- We have time scale separations :

$$\text{turnover time} = 1 \ll 1/\alpha = \text{forcing or dissipation time}$$

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Relation with the Stationary States of Euler Eq.

$$\frac{\partial \omega}{\partial t} + \mathbf{u}.\nabla\omega = \nu\Delta\omega - \alpha\omega + \sqrt{2\alpha}f_s \quad (3)$$

- Time scale separation : Magenta terms are small.
- At first order, the dynamics is nearly a 2D Euler dynamics. The flow self organizes and converges towards stationary solutions for Euler equations :

$$\mathbf{u}.\nabla\omega = 0 \text{ or equivalently } \omega = f(\psi)$$

where the Stream Function ψ is given by : $\mathbf{u} = \mathbf{e}_z \times \boldsymbol{\nabla}\psi$

- Stationary flows of Euler equation will play a crucial role. Degeneracy : what does select f ?

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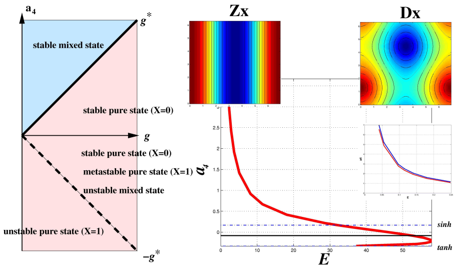
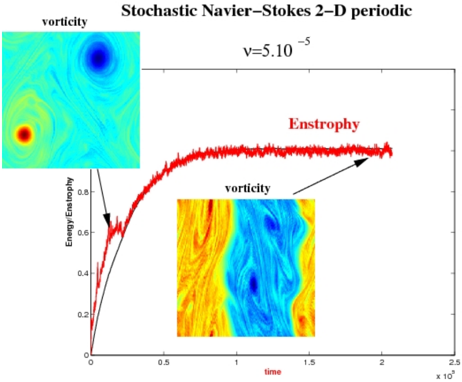
Stationary Flows for the 2D-Euler Eq. (doubly periodic)



Bifurcation analysis : degeneracy removal, either by the domain geometry (g) or by the nonlinearity of the vorticity-stream function relation (f, parameter a_4)

General degeneracy removal mechanism. Prediction of similar phase diagram for a large class of models (QG, SW)

Numerical Simulation of the 2D Stochastic NS Eq.



Very long relaxation times. 10^5 turnover times

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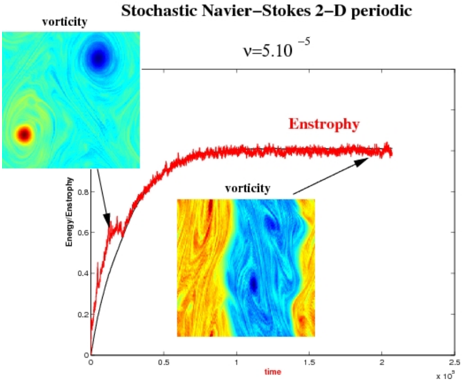

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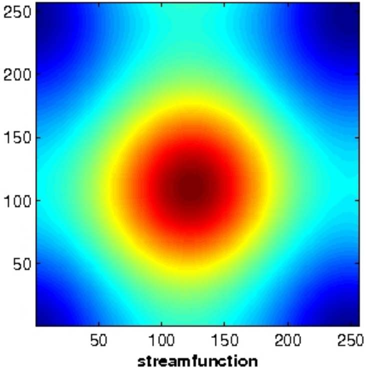
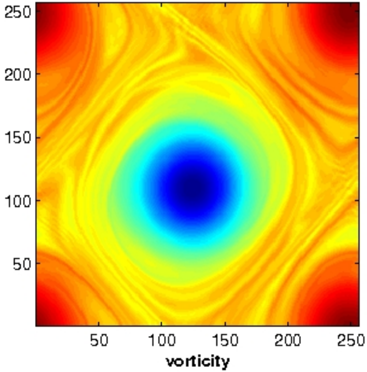
Out of Equilibrium Stationary States : Dipoles





Are we close to some stationary flows of Euler Eq. ?

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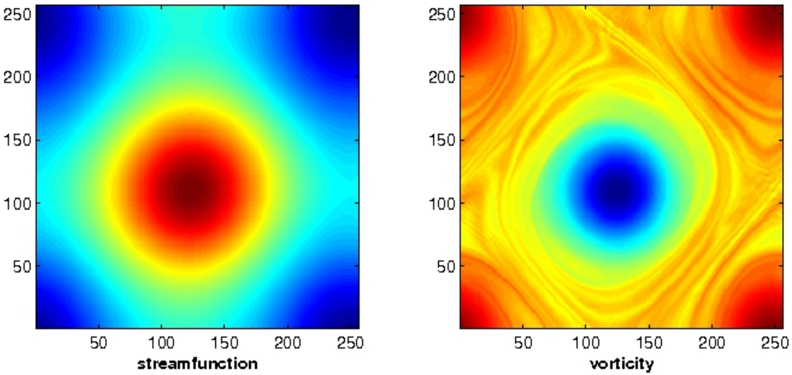

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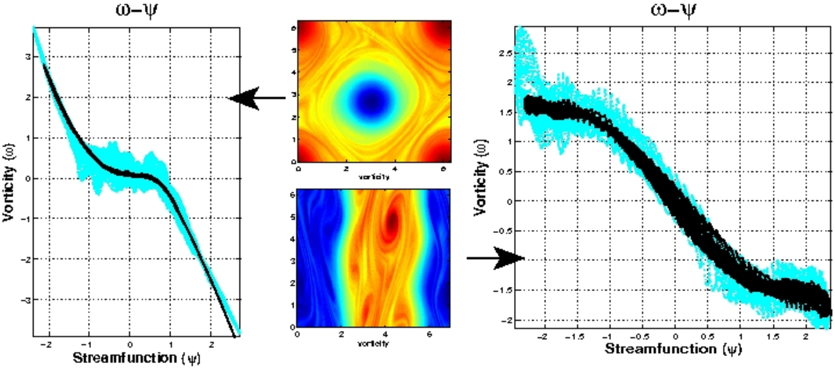
Vorticity-Streamfunction Relation



Conclusion : we are close to stationary flows of Euler Eq.

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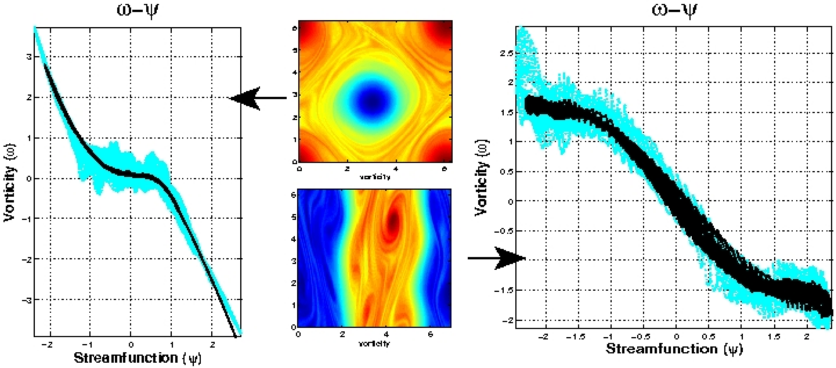

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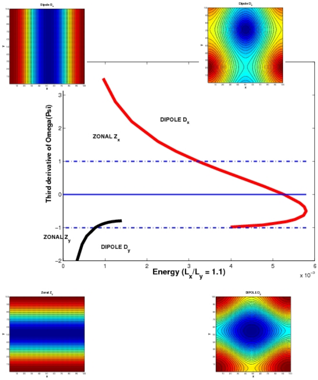
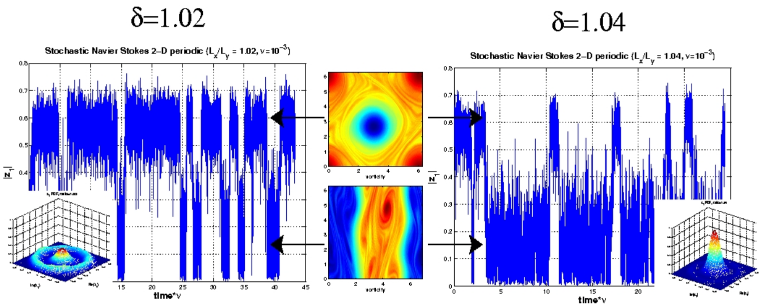
Statistical Equilibria for the 2D-Euler Equation with Periodic Boundary Conditions



A second order phase transition

Out of Equilibrium Phase Transition

The time series and PDF of the Order Parameter



Order parameter : $z_1 = \int dxdy\ \exp(iy)\omega(x, y)$.

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Experimental Applications

- Using the equilibrium theory, we can predict the existence of out of equilibrium phase transitions
- Or phase transitions governed by the domain geometry, by the topography, by the energy
- Prediction of flow topology change in Quasi-Geostrophic and Shallow Water dynamics (rotating tank experiments)

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A Theory for Large Scales of the 2D Navier-Stokes Eq.

An Adiabatic Reduction ?

- Euler's equations have an infinity of stationary solutions : $\omega = f(\psi)$ for any f (Euler's equations have degenerate equilibria.)
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- For Navier-Stokes with weak forces, what does select f
- Time scale separation. An adiabatic reduction ?

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- Euler's equations have an infinity of stationary solutions : $\omega = f(\psi)$ for any f (Euler's equations have degenerate equilibria.)
- For the conservative dynamics, equilibrium statistical mechanics selects f
- For Navier-Stokes with weak forces, what does select f
- Time scale separation. An adiabatic reduction ?

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Adiabatic Reduction for the Stochastic-NS Eq.

First step (current work) : the stochastic linearised Navier Stokes equation

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- The linear operator is non normal (no mode decomposition) ! Further difficulties
- The second step : An equation that describes only the large scale evolution

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The 2D Linearized Stochastic Euler Equation

- A stable equilibria for the 2D Euler equation $\mathbf{v}_0$, with vorticity q_0 : $\mathbf{v}_0.\nabla q_0 = 0$.
- The 2D Euler equation, linearized close to $\mathbf{u}_0$, with stochastic forces :

$$dq + \mathbf{v}.\nabla q_0 dt + \mathbf{v}_0.\nabla q dt = -\alpha q dt + \sqrt{\sigma} \sum_{kl} f_{kl} \tilde{e}_{kl} dW_{kl}(t)$$

- An infinite dimensional Orstein-Ulhenbeck process (Gaussian, two point correlations, Lyapounov equation)
- Theoretical difficulty : the deterministic linearized operator is non normal (no mode decomposition)
- Landau damping or Orr mechanism

The 2D Linearized Stochastic Euler Eq.

Stochastic Landau damping (F. B.)

- Resonance for the vorticity autocorrelation function for small α : $\langle q(r,0)q(r',0)\rangle_S = \mathcal{O}(\sigma/\alpha) \propto_{\alpha\to 0} \delta(r-r')$
- No resonances for the stream function and velocity : $\langle \psi(r,0)\psi(r',0)\rangle_S = \mathcal{O}(\sigma)$ and $\langle \mathbf{v}(r,0)\mathbf{v}(r',0)\rangle = \mathcal{O}(\sigma)$
- In the small dissipation limit, the velocity stochastic process has a definite limit (Stochastic Landau damping)

Further issues :

1. Numerical resolution of the Lyapounov equation (more physical issues) (F. Gallaire)
2. Is an adiabatic reduction possible ?

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Summary

Messages :

- We can predict and observe out of equilibrium phase transitions for the 2D-Stochastic Navier Stokes equation
- We propose experiments to observe such phenomena (Navier Stokes, Quasi Geostrophic, or Shallow Water dynamics)
- Theory for the 2D stochastic linearized Euler equation. Stochastic Landau damping.

Other related recent results :

- Simplified variational problems for the statistical equilibria of 2D flows. F. Bouchet, Physica D, 2007
- Phase transitions, ensemble inequivalence and Fofonoff flows. A. Venaille and F. Bouchet, sub. to Phys. Rev. Lett.

Outline

1. Motivations
   - Geophysical flows and statistical physics
2. Statistics of the large scales of 2D turbulent flows
   - Classical views for 2D turbulence, inverse energy cascade or equilibrium stat. mech. ?
   - Out of equilibrium phase transitions in the 2D Stochastic Navier-Stokes Eq. (E. Simonnet, H. Morita and F. B.)
3. The 2D linearized Euler Eq. with stochastic forces
4. Conclusions
   - More

The Stochastic Forces

$$\frac{\partial \omega}{\partial t} + \mathbf{u}.\nabla \omega = \nu \Delta \omega - \alpha \omega + \sqrt{2\alpha} f_s \quad (4)$$

$$f_S(\mathbf{x}, t) = \sum_{\mathbf{k}} f_{\mathbf{k}} \eta_{\mathbf{k}}(t) e_{\mathbf{k}}(\mathbf{x}) \quad (5)$$

where the $e_{\mathbf{k}}$'s are the Fourier modes (Laplacian eigenmodes) and $< \eta_{\mathbf{k}}(t) \eta_{\mathbf{k}'}(t') >= \delta_{\mathbf{k},\mathbf{k}'} \delta(t - t')$ (white in time)

For instance $f_{\mathbf{k}} = A \exp - \frac{(|\mathbf{k}| - m)^2}{2\sigma^2}$ with $\frac{1}{2} \sum \frac{|f_{\mathbf{k}}|^2}{|\mathbf{k}|^2} = 1$ (smooth in space).

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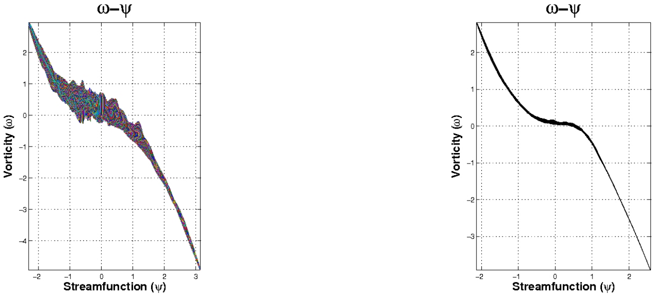
The 2D Stochastic Navier-Stokes Equation

$$\frac{\partial \omega}{\partial t} + \mathbf{u}.\nabla\omega = \nu\Delta\omega + \sqrt{\nu} f_s$$

- Some recent mathematical results : Kuksin, Sinai, Shirikyan, Bricmont, Kupianen, etc
  - Existence of a stationary measure μ_ν. Existence of $lim_{\nu \to 0}\ \mu_\nu$
  - In this limit, almost all trajectories are solutions of the Euler equation
- We would like to obtain more physical results :
  - What is the link of this limit $\nu \to 0$ with the RSM theory ?
  - Will we stay close to some stationary solutions of the Euler equation ?
  - Can we describe these stationary states and their properties ?

Vorticity-Streamfunction Relation (dipole)

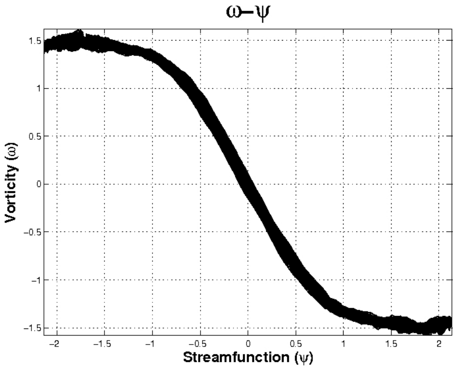


Snapshot

With time averaging

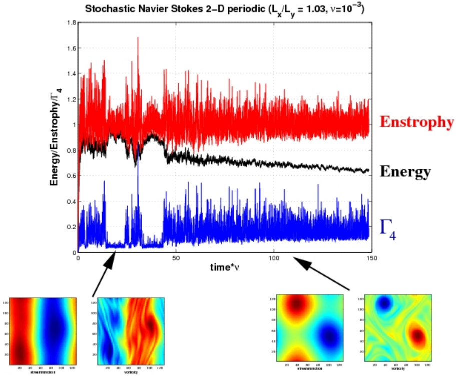
Conclusion : we are close to some statistical equilibria

Vorticity-Streamfunction Relation (Zonal Flow)



Discrepancies with statistical equilibrium ?

An Out of Equilibrium Phase Transition

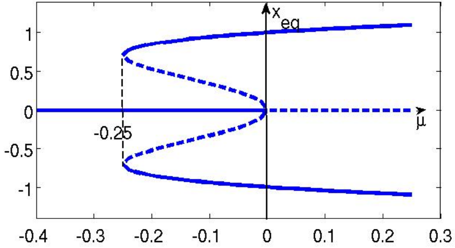


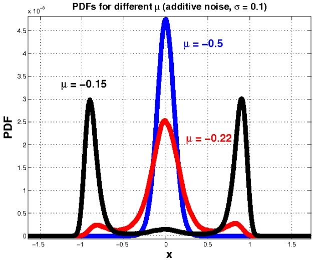
Localisation of the phase transition

An Analogy with a Subcritical Bifurcation ?

A subcritical bifurcation perturbed by an additive noise :

$$dx = x\left(\mu + x^2 - x^4\right)dt + \sqrt{\sigma}dW_t$$



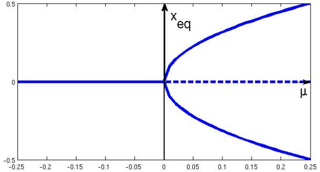


Deterministic bifurcation diagram

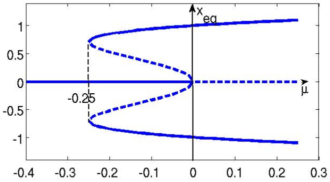
PDFs

Incompatible with the Equilibrium Phase Transition ?

At equilibrium we observe a second order phase transition (symmetry breaking)



Pitchfork bifurcation diagram (equilibrium)



Bifurcation diagram for the SDE model