Fluctuations relations for diffusion processes

UMR 5672 CNRS - ENS Lyon

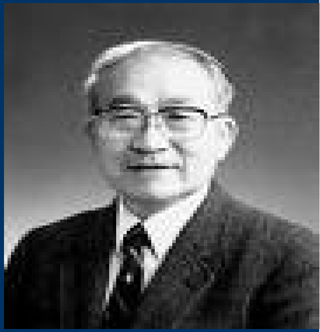
Search for a common ground between some recent ideas in non-equilibrium statistical mechanics and in turbulence

Weakly non-equilibrium dynamics

Fluctuation dissipation-theorem (1951)

Herbert B.Callen(1919-1993)

Green-Kubo relations (~1950)



Ryogo **KUBO** (1920 **...**)

Onsager relations (1931)

Lars Onsager (1903-1976)

Far from equilibrium dynamics

FLUCTUATION RELATIONS

Jarzynski relations (1997)

Evans-Searle relations(1994)

Searle | Ponti Knit | **Fashion**

Gallavotti-Cohen relations(1995)

Fluctuation relations for diffusion processes. R. Chetrite, K.Gawedzki,

to be appear in CMP

Toy model 1 : Dissipative Langevin dynamics

$$\frac{dx}{dt} = -\Gamma \nabla H + \eta(t)$$

$$\langle \eta_t^i \eta_{t'}^j \rangle = \frac{2}{\beta} \Gamma^{ij} \delta(t - t')$$

Einstein relation

The Gibbs density $\exp(-\beta H(x))$ is the invariant density. The density current of this density vanishes and we have the **detailled balance (DB)**

EQUILIBRIUM STATE

To Enter in the non-equilibrium world
BREAK :

- Stationnarity : H → H_t
- Hamiltonian form : add non-gradiant term, no local term → **TURBULENCE**

- Einstein relation : $\langle \eta_i^a(t)\, \eta_j^b(t') \rangle = 2\gamma_i \beta_i^{-1} \delta_{ij} \delta^{ab} \delta(t-t')$

→ Fourier law

General mathematical setup for our work

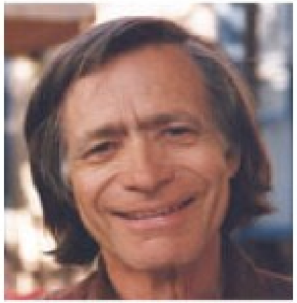
Vincent Doeblin, soldat téléphoniste, automne 1939

$$\frac{dx}{dt} = u_t(x) + v_t(x)$$

$$\left\langle v_t^i(x) v_{t'}^j(x') \right\rangle = \delta(t-t') D_t^{ij}(x,x')$$

ITO (1915-)

Exemples of diffusive equations :



KRAICHNAN

1) Kraichnan model of turbulent flow with $u_t \equiv 0$

2) Deterministic dynamical system with $v_t \equiv 0$

3) Langevin dynamics

A tribute For late Robert Kraichann.

Passive transport of particles:

- Lagrangian tracers with no inertia:

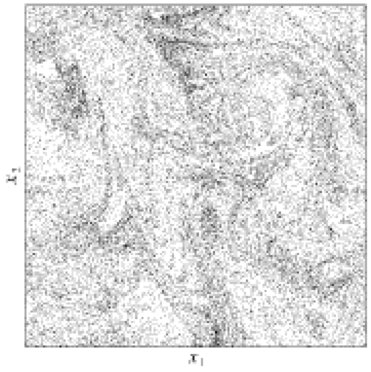
$$\dot{\boldsymbol{r}} = \boldsymbol{v}_t(\boldsymbol{r})$$

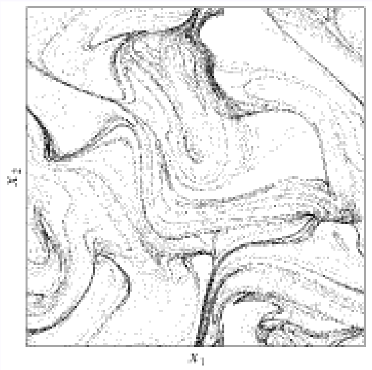
- particles with inertia:

$$\dot{\boldsymbol{r}} = \boldsymbol{v}, \qquad \dot{\boldsymbol{v}} = -\frac{1}{\tau}\left(\boldsymbol{v} - \boldsymbol{v}_t(\boldsymbol{r})\right)$$

friction force

Stokes time





from **J. Bec**, J. Fluid Mech. 528, 255-277 (2005)

transport in **Kraichnan velocities**: Gaussian random ensemble of fields $\boldsymbol{v}_t(\boldsymbol{r})$ decorrelated in time widely used in last years to model turbulent phenomena

Krzysztof shown in the last talk that we can write :

For a general diffusive system, the **DB** or **MDB** may be replaced by the **detailed fluctuation relation (DFR)** :

$$\mu_0(dx)\ P_{0,T}(x;dy|W)\ e^{-W} = \mu'_0(dy^*)\ P'_{0,T}(y^*;dx^*|-W) \quad \textbf{(DFR)}$$

where

- $\mu_0(dx) = e^{-\varphi_0(x)}dx$ is the initial distribution of the forward process
- $\mu'_0(dy^*) = e^{-\varphi'_0(y^*)}dy^*$ is the initial distribution of the **backward process**
- $P_{0,T}(x,dy|W)$ is the transition probability with the constraint $\mathcal{W}=W$ fixing the value of a functional $\mathcal{W}$ of the forward process with the interpretation of the entropy production
- $P'_{0,T}(y^*,dx^*|W)$ is the similar constraint transition probability for the backward process

For the *tangent process* , we have the *multiplicative large deviation* form :

$$P^T(x \to y, \vec{\rho})dy \propto \exp(-TZ(\frac{\vec{\rho}}{T}))dy$$

and

The **generalized Gallavotti-Cohen** relation :

$$Z(\frac{\vec{\rho}}{T}) - \sum \frac{\rho_i}{T} = Z^r(-\frac{\overleftarrow{\rho}}{T})$$

$Z(\vec{\sigma})$ is important for turbulent transport since it determines:

- rate of decay of moments of transported scalar
- rate of growth of density and magnetic field fluctuations
- multi-fractal dimensions of attractor for tracers in compressible flows and for inertial particles
- polymer stretching in presence of turbulence

KRAICHNAN CASE

$$Z\left(\frac{\vec{\rho}}{T}\right) - \sum \frac{\rho_i}{T} = Z\left(-\frac{\vec{\rho}}{T}\right)$$

Z is accessible analytically in the Kraichnan model of turbulent advection via relations to integrable models

Chetrite-Dellanoy-Gawedzki, J.Stat.Phys 2006

- In the homogeneous isotropic case with $D^{ab}(\boldsymbol{r} - \boldsymbol{r}')$ rotationally covariant, $\sigma_a(t)$ satisfy the Langevin equation of the **Calogero-Sutherland** type
- In the homogeneous $2d$ case with square symmetry, $Z\left(\frac{\vec{\rho}}{T}\right)$ is expressed by the ground state energy of the **Lamé-Hermite** elliptic Hamiltonian
- In the $1d$ homogeneous case, for the inertial particles $\delta R(t)\, \mathrm{e}^{t/(2\tau)} \equiv \psi(t)$ behaves as the wave function $\psi(x)$ in $1d$ **Anderson** localization in a δ-correlated potential

For the homogeneous isotropic flow

$$-\nabla_c \nabla_d D^{ab}(\mathbf{0}) = \beta \left(\delta^a_c \delta^b_d + \delta^a_d \delta^b_c \right) + \gamma\, \delta^{ab} \delta_{cd}$$

$$Z\left(\frac{\boldsymbol{\sigma}}{t}\right) = \frac{1}{2(\beta+\gamma)} \left[\sum_a \left(\frac{\sigma_a}{t} - \lambda_a \right)^2 - \frac{\beta}{(d+1)\beta+\gamma} \left(\sum_a \left(\frac{\sigma_a}{t} - \lambda_a \right) \right)^2 \right]$$

with equally spaced **Lyapunov exponents**:

$$\lambda_a = \frac{\beta+\gamma}{2}(d - 2a + 1) - \frac{(d+1)\beta+\gamma}{2}$$

For the homogeneous $2d$ flow on aperiodic square

$$-\nabla_c \nabla_d D^{ab}(\mathbf{0}) = 2\alpha\, \delta^{ij}_{kl} + \beta \left(\delta^a_c \delta^b_d + \delta^a_d \delta^b_c \right) + \gamma\, \delta^{ab} \delta_{cd}$$

$$Z\left(\frac{\sigma_1}{t}, \frac{\sigma_2}{t}\right) = \frac{\left(\frac{\sigma_1+\sigma_2}{t} + 2\alpha + 3\beta + \gamma\right)^2}{4(2\alpha+3\beta+\gamma)}$$

$$+ \max_{\mu} \left[\mu \left(\frac{\sigma_1 - \sigma_2}{t} \right) - (\beta - \gamma)\mu(\mu+1) + 2\, E_\mu \right]$$

where E_μ is the ground state energy of the periodic 1-dimensional **Schrödinger** operator of the **Lamé-Hermite** type:

$$-\frac{d^2}{du^2} + \mu(\mu+1)\, V(\phi(u))$$

with the attractive periodic potential

$$V(\phi) = -\frac{\alpha(\alpha+\gamma)}{\gamma + \alpha \sin^2 \phi}, \qquad u(\phi) = \int_0^{\phi} \left[\gamma + \alpha \sin^2 \varphi\right]^{-1/2} d\varphi$$