Singularités complexes des équations d'Euler et Navier–Stokes

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Main motivation

Do 3D incompressible flows governed by the Euler or the Navier–Stokes equation

$$\begin{aligned} \partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} &= -\nabla p + \nu\Delta\boldsymbol{v}, \\ \nabla\cdot\boldsymbol{v} &= 0, \end{aligned}$$

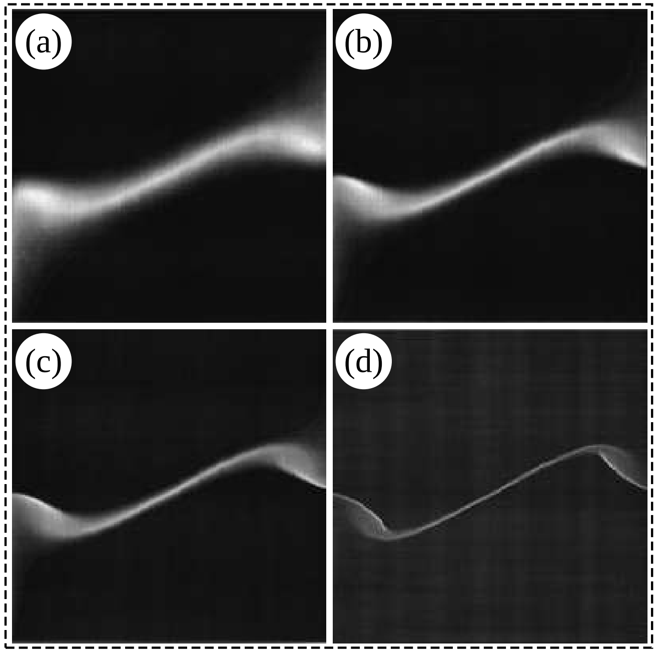
with *smooth* (e.g. analytic) initial data develop finite-time singularities (blowup)?

Why study singularities?

- Dissipative anomaly in turbulence: are singularities responsible for non-vanishing dissipation as $\nu \to 0$?

- Depletion of nonlinearity for the Euler equation

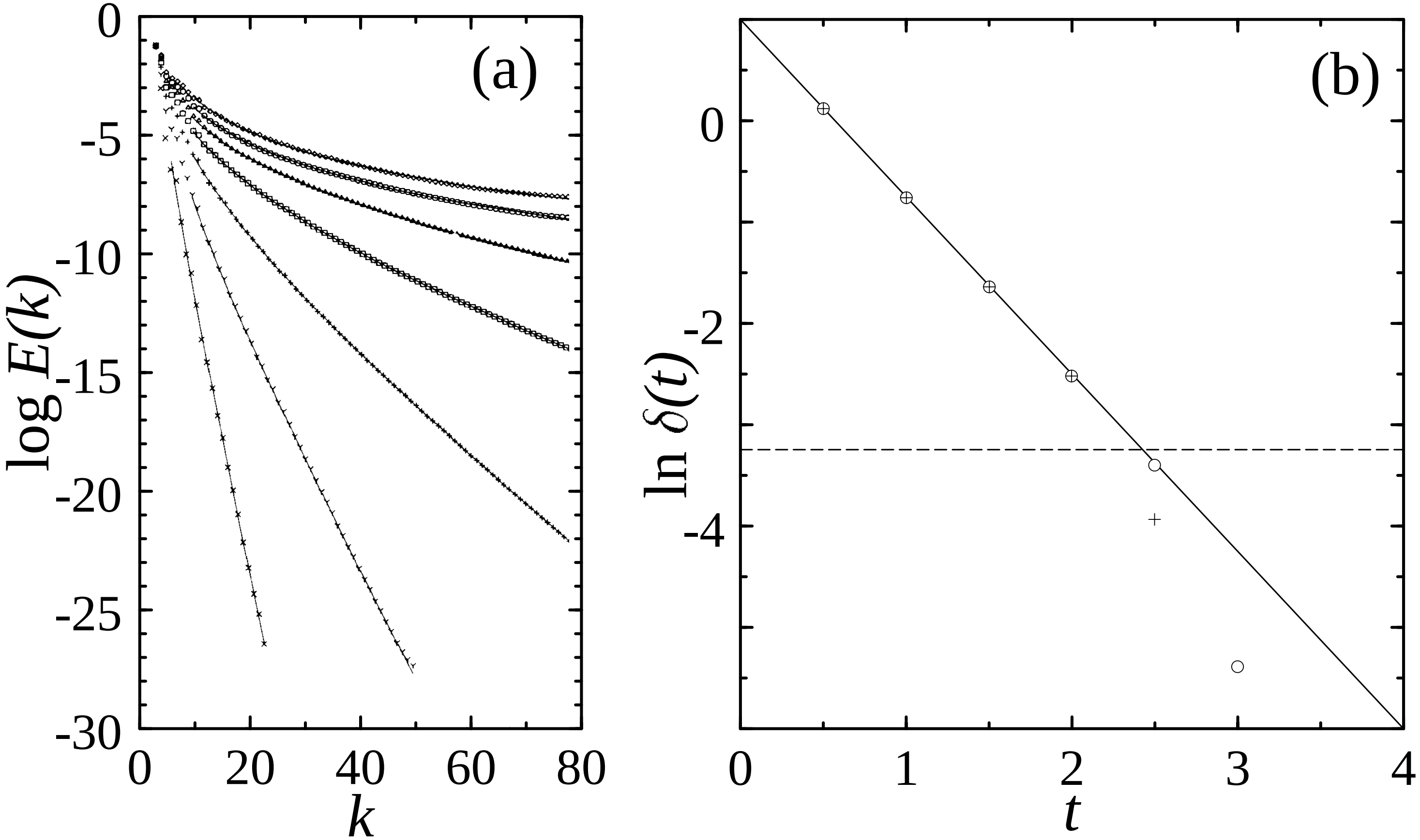
- Numerics: in general solutions are tamer than predicted analytically



- Strong intermittency in the dissipation range: signature of singularities?

Why complex singularities?

- Bardos and Benachour (1977) *proved* that in 3D, for analytic initial data, any hypothetical real-space singularity at $t_\star$ is preceded by complex-space ($\mathbb{C}^3$) singularities within a distance $\delta(t)$ of $\mathbb{R}^3$ which vanishes as $t \to t_\star$.
- Numerical monitoring of complex singularities: exponential fall-off $2\delta(t)$ of energy spectra



Series expansion for the 2D Navier–Stokes equation (Sinai 2005)

- The two-dimensional Navier–Stokes equation

$$\partial_t \nabla^2 \psi - J(\psi, \nabla^2 \psi) = \nu \nabla^2 \big(\nabla^2 \psi \big).$$

is written in Fourier space as

$$\hat{\psi}(k,t) = \hat{\psi}(k,0) e^{-\nu |k|^2 t} - \frac{1}{|k|^2} e^{-\nu |k|^2 t} \int_0^t e^{\nu |k|^2 s} \Big(\sum_{l+l'=k} l \wedge l' |l'|^2 \hat{\psi}(l,s) \hat{\psi}(l',s) \Big)\, ds.$$

- Initial conditions with initial modes $\boldsymbol{p}$ and $\boldsymbol{q}$

$$\psi_0(z_1, z_2) = \hat{F}(\boldsymbol{p}) e^{-i\boldsymbol{p}\cdot\boldsymbol{z}} + \hat{F}(\boldsymbol{q}) e^{-i\boldsymbol{q}\cdot\boldsymbol{z}} + \text{c.c.}$$

- Putting $\hat{F}(\boldsymbol{p})$, $\hat{F}(\boldsymbol{q}) \to A\hat{F}(\boldsymbol{p}), A\hat{F}(\boldsymbol{q})$ and taking $A \to 0$ we recover solutions starting with initial conditions

$$\psi_0(z_1, z_2) = \hat{F}(\boldsymbol{p}) e^{-i\boldsymbol{p}\cdot\boldsymbol{z}} + \hat{F}(\boldsymbol{q}) e^{-i\boldsymbol{q}\cdot\boldsymbol{z}}$$

- $\nu \to 0$ yields the corresponding solutions of the Euler equation
- These solutions have the self-similar form

$$\psi(z_1, z_2) = \frac{1}{t} F(\tilde{z}_1, \tilde{z}_2), \qquad (\tilde{z}_1, \tilde{z}_2) = (z_1 + i\,\lambda_1 \ln t,\; z_2 + i\,\lambda_2 \ln t),$$

Short-time asymptotics for the 2D Euler equation

- Complexified 2D Euler equation $(z_1, z_2) = (x_1 + iy_1, x_2 + iy_2)$

$$\partial_t \nabla^2 \psi = J(\psi, \nabla^2 \psi)$$

- Short-time asymptotic régime: initial condition with two basic modes $\boldsymbol{p}$ and $\boldsymbol{q}$

$$\psi_0(z_1, z_2) = \hat{F}(\boldsymbol{p}) e^{-i\boldsymbol{p}\cdot\boldsymbol{z}} + \hat{F}(\boldsymbol{q}) e^{-i\boldsymbol{q}\cdot\boldsymbol{z}}$$

- Solution can be written in terms of

$$G_{(\boldsymbol{p},\boldsymbol{q})}(\xi_1, \xi_2) = e^{\xi_1} + e^{\xi_2} + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \hat{G}_{(\boldsymbol{p},\boldsymbol{q})}(k_1, k_2) e^{k_1 \xi_1} e^{k_2 \xi_2}.$$

- Coefficients $\hat{G}_{(\boldsymbol{p},\boldsymbol{q})}(k_1, k_2)$ satisfy a certain recursion relation.

- In principle, all coefficients can be calculated symbolically, e.g.

$$\hat{G}_{(\boldsymbol{p},\boldsymbol{q})}(1,1) = -\frac{|\boldsymbol{q}|^2 - |\boldsymbol{p}|^2}{|\boldsymbol{p}+\boldsymbol{q}|^2}, \qquad \hat{G}_{(\boldsymbol{p},\boldsymbol{q})}(2,1) = \frac{1}{2} \frac{|\boldsymbol{q}|^2 - |\boldsymbol{p}|^2}{|\boldsymbol{p}+\boldsymbol{q}|^2} \frac{|\boldsymbol{p}+\boldsymbol{q}|^2 - |\boldsymbol{p}|^2}{|2\boldsymbol{p}+\boldsymbol{q}|^2}$$

- Numerics indicate that $(-1)^{k_1} \hat{G}_{(\boldsymbol{p},\boldsymbol{q})}(k_1, k_2) \geq 0$ for all (k_1, k_2) except for $(1, 0)$

Pseudohydrodynamics

- Function $G_{(\boldsymbol{p},\boldsymbol{q})}(\xi_1,\xi_2)$ is solution of

$$\Delta_{(\boldsymbol{p},\boldsymbol{q})}G = J(H_{(\boldsymbol{p},\boldsymbol{q})},\Delta_{(\boldsymbol{p},\boldsymbol{q})}G),$$

where $H_{(\boldsymbol{p},\boldsymbol{q})}(\xi_1,\xi_2) = G_{(\boldsymbol{p},\boldsymbol{q})} + \xi_1 - \xi_2$ and $\Delta_{(\boldsymbol{p},\boldsymbol{q})}$ is a modified Laplacian

$$\Delta_{(\boldsymbol{p},\boldsymbol{q})} = |\boldsymbol{p}|^2\frac{\partial^2}{\partial\xi_1^2} + 2\boldsymbol{p}\cdot\boldsymbol{q}\,\frac{\partial}{\partial\xi_1}\frac{\partial}{\partial\xi_2} + |\boldsymbol{q}|^2\frac{\partial^2}{\partial\xi_2^2}$$

- Relevant parameters:
  - Ratio of moduli $\eta = |\boldsymbol{q}|/|\boldsymbol{p}|$ of the basic vectors $\boldsymbol{p}$ and $\boldsymbol{q}$
  - Angle ϕ between $\boldsymbol{p}$ and $\boldsymbol{q}$
- Trivial solution for $\eta = 1$

$$G(\xi_1,\xi_2) = e^{\xi_1} + e^{\xi_2}, \qquad H(\xi_1,\xi_2) = e^{\xi_1} + e^{\xi_2} + \xi_1 - \xi_2$$

- Perturbative expansion in $(\eta - 1)$ yieds linearised Euler equation with a source term at first subleading order
- For the linearised Euler equation the vorticity diverges near the singularities as s^{-1}, where s is the distance to the singularities

Geometry and nature of sigularities: numerical results

- Asymptotics of $\hat{G}_{(\boldsymbol{p},\boldsymbol{q})}(k_1, k_2)$ in polar coordinates $\boldsymbol{k} = |\boldsymbol{k}|(\cos\theta, \sin\theta)$

$$\hat{G}_{(\boldsymbol{p},\boldsymbol{q})}(|\boldsymbol{k}|, \theta) \simeq C_{(\boldsymbol{p},\boldsymbol{q})}(\theta)|\boldsymbol{k}|^{-\alpha_{(\boldsymbol{p},\boldsymbol{q})}} e^{-\delta_{(\boldsymbol{p},\boldsymbol{q})}(\theta)|\boldsymbol{k}|}, \qquad \text{for } |\boldsymbol{k}| \to \infty$$

- Vorticity diverges near the singularities

$$\omega \sim s^{-\beta},$$

where $\alpha + \beta = 7/2$

- High-precision numerical calculation: exponent α (and thus β) depends on ϕ (but not on η)

- Conjecture:

$$\text{exponent } \alpha(\phi) \text{ increases monotonically from } \alpha(0) = 5/2 \text{ to } \alpha(\pi) = 3$$

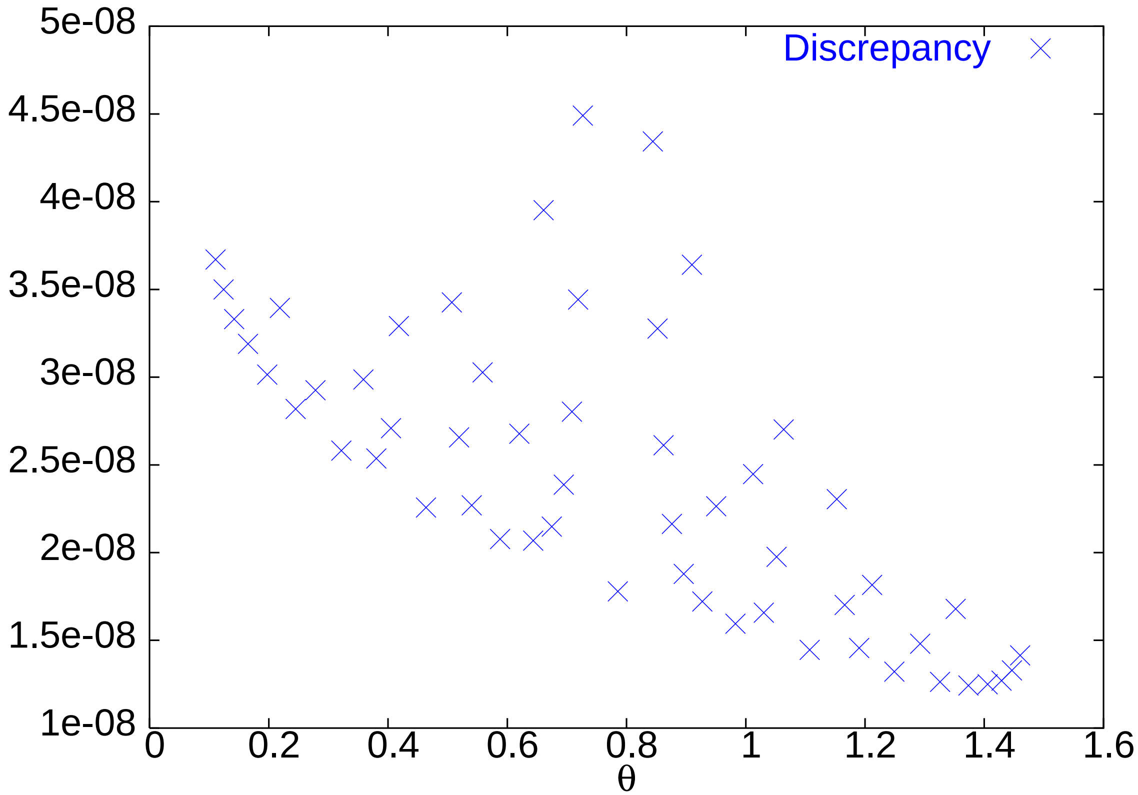
$$\Updownarrow$$

$$\beta(\phi) \text{ decreases from } \beta(0) = 1 \text{ to } \beta(\pi) = 1/2$$

Case $\phi = 0$: precise determiation of the nature of singularities

- Numerically determined asymptotic expansion obtained using an asymptotic interpolation procedure

$$\hat{G}(|\boldsymbol{k}|, \theta) \simeq C(\theta)|\boldsymbol{k}|^{-\frac{5}{2}} e^{-\delta(\theta)|\boldsymbol{k}|} \left[1 + \frac{b_1(\theta)}{|\boldsymbol{k}|} + \frac{a_2(\theta)\ln|\boldsymbol{k}|}{|\boldsymbol{k}|^2} + O\left(\frac{1}{|\boldsymbol{k}|^2}\right)\right]$$



- Theory in progress

Case $\phi = \pi$: work in progress

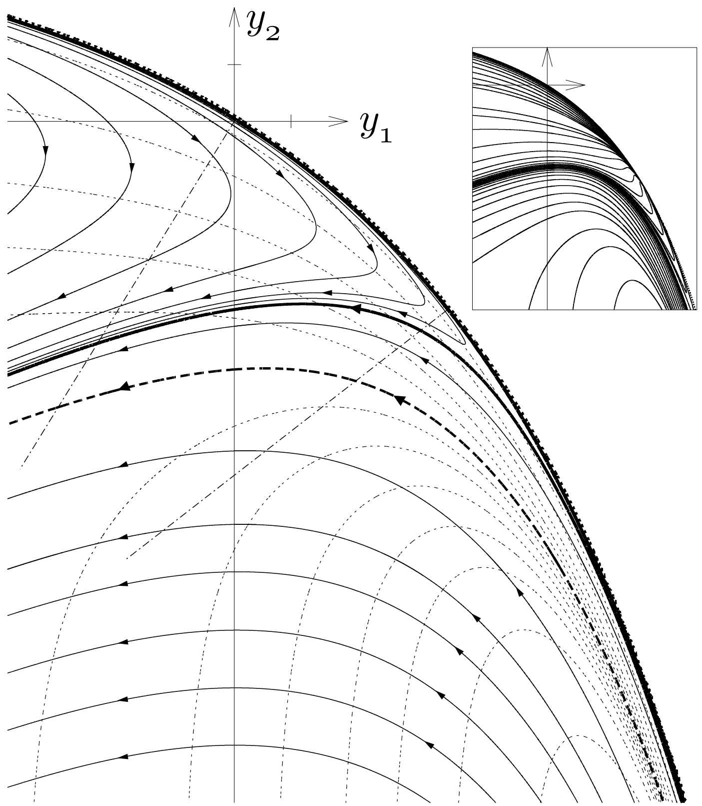
- Resonances: vanishing denominator in the coefficients $\hat{G}_{(\boldsymbol{p},\boldsymbol{q})}(k_1, k_2)$
  - Fix angle $\phi = \pi$ first: poles at values $\eta = 1, 2, 3, ...$, e.g.

$$\hat{G}(2,1) = \frac{1}{2}\frac{\eta+1}{\eta-1}\frac{\eta}{\eta-2}$$

  - Fix $\eta = 1, 2, 3, ...$ first and then take the limit $\phi \to \pi$: all coefficiets have finite values
- Coefficients $\hat{G}(k_1, k_2)$ are not continuous for $\phi = \pi$ and $\eta = 1, 2, 3, ...$
- Numerical study of the limit $\phi \to \pi$
- Conjectured value of the exponent $\alpha = 3$

Depletion of non-linearity for the 2D Euler equation

- Nature of singularities
  - Case $\phi = 0$: well rendered by the liearised Euler equation
  - Case $\phi = \pi$: the same as for the 2D Burgers equation
- The degree of non-linearity is determined by the parameter ϕ
- Geometry of the flow



Short-time asymptotics for 3D Euler equation

- Initial conditions:
  - Kida–Pelz flow

$$
\begin{aligned}
v_1(z_1, z_2, z_3) &= \sin z_1 (\cos 3z_2 \cos z_3 - \cos z_2 \cos 3z_3) \\
v_2(z_1, z_2, z_3) &= \sin z_2 (\cos 3z_3 \cos z_1 - \cos z_3 \cos 3z_1) \\
v_3(z_1, z_2, z_3) &= \sin z_3 (\cos 3z_1 \cos z_2 - \cos z_1 \cos 3z_2)
\end{aligned}
$$

  - Permutation (Pelz–Ohkitani) flow

$$
\begin{aligned}
v_1(z_1, z_2, z_3) &= \sin z_2 + \sin z_3 \\
v_2(z_1, z_2, z_3) &= \sin z_1 + \sin z_3 \\
v_3(z_1, z_2, z_3) &= \sin z_1 + \sin z_2
\end{aligned}
$$

- Numerics indicate that in 3D singularities are non-universal as well

Conclusions and future work

- Nature of complex singularities for inviscid flows in 2D and 3D depends on the initial conditions and is thus non-universal
- Theory in progress in 2D
- Relaxed multiplication algorithms are being implemented using “Mathemagix” for more efficient calculation of solutions
- Detailed analysis of the 3D solutions is envisaged
- Extension to the viscous case: nature and geometry of the singularities of solutions of the Navier–Stokes equation