3D–Navier–Stokes Equations in bounded domains : the challenging question of the zero viscosity limit, and the choice of boundary conditions

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Motivation : three situations without any boundary layer for small t ...
a specific structure for viscous solutions and the continuity on the viscosity parameter

(!) The difficult case when the viscous solutions are required to satisfy the standard no-slip boundary condition : Dirichlet's identity, preserved in the class of strong solutions when $\nu \to 0$, is not generally satisfied by the solution to the Euler model.

(?) $\ldots$, the choice of appropriate slip b. c. for viscous extension of Euler's solution
(?) $\ldots$, *parallely the existence of associated viscous solutions must be proved*

Data

Ω $\ldots$ a general regular bounded domain in $\mathbb{R}^3$, locally on one side of its boundary $\partial\Omega$ (say a impermeable surface of the class $C^{3,1}$), and let $T > 0$, $\mathbf{f}$, $\mathbf{u}^*$, $\ldots$, and the Euler initial–boundary value problem

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \mathbf{f} \qquad \text{in } Q_T := \Omega \times (0, T), \tag{1}$$

$$\operatorname{div} \mathbf{u} = 0 \qquad \text{in } Q_T, \tag{2}$$

$$\mathbf{u}(.\,, 0) = \mathbf{u}^* \qquad \text{in } \Omega, \tag{3}$$

$$\mathbf{u} \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma_T := \partial\Omega \times (0, T). \tag{4}$$

The local in time existence of a strong solution to the Euler model is a classical result (T. Kato (1972), J. P. Bourguignon & H. Brezis (1974), R. Temam (1975))

Suppose that $r > 0$, $\mathbf{u}^* \in \mathbf{W}^{5/2+r,2}(\Omega) \cap \mathbf{L}^2_\sigma(\Omega)$ *and* $\mathbf{f} \in L^1\big(0, T;\ \mathbf{W}^{5/2+r,2}(\Omega)\big)$. *Then there exists* $T_0 \in (0, T]$ *and a unique solution* $\mathbf{u}^0$ *to the Euler model (1)–(4) on the time interval* $(0, T_0)$ *such that*
$\mathbf{u}^0 \in L^\infty\big(0, T_0; \mathbf{W}^{5/2+r,2}(\Omega)\big)$, $\partial_t \mathbf{u}^0 \in L^1\big(0, T_0; \mathbf{W}^{5/2+r,2}(\Omega)\big) + L^\infty\big(0, T_0; \mathbf{W}^{r,2}(\Omega)\big)$.
p^0 *is the associated pressure.* Our choice will be $r = 3/2$.

Is this solution $\mathbf{u}^0$ of the Euler model a limit for $\nu \to 0+$
of a (possibly unique) branch of (strong) solutions $\mathbf{u}^\nu$
of an appropriate Navier–Stokes model ? The question makes sense for $T = T_0$,

and the problem we want to treat is
to find, in addition to the no–flux condition (8),
a complementary slip boundary condition (*) for velocity (or for vorticity),
and a structure for these solutions such that
the branch is unique and continuous in dependence on ν *in some interval* $[0, \nu^*)$.

An appropriate Navier–Stokes model is given by the equation

$$\partial_t \mathbf{u}^\nu + (\mathbf{u}^\nu \cdot \nabla)\mathbf{u}^\nu = -\nabla p + \nu \Delta \mathbf{u}^\nu + \mathbf{f} \qquad \text{in } Q_{T_0} \tag{5}$$

and by the conditions

$$\operatorname{div} \mathbf{u}^\nu = 0 \qquad \text{in } Q_{T_0}, \tag{6}$$

$$\mathbf{u}^\nu(.\,,0) = \mathbf{u}^* \qquad \text{in } \Omega, \tag{7}$$

$$\mathbf{u}^\nu \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma_{T_0}, \tag{8}$$

and completed by an appropriate additional slip boundary condition (*).

Three choices for (*) . . .

Three choices for (*) on Γ_{T_0}
with appropriate inhomogeneous ”data”, resp. $\mathbf{a}^\nu$, $b^{k,\nu}$, $\mathbf{c}^{\nu,\gamma}$ are :

” inhomog. vorticity-type conditions or generalized impermeability conditions ”

$$\mathbf{curl}^{\,k}\mathbf{u}^\nu \cdot \mathbf{n} = b^{k,\nu} \text{ with } b^{k,\nu} := \mathbf{curl}^{\,k}(\mathbf{u}^0 + \nu\mathbf{v}^0)\cdot\mathbf{n} \;\; k = 1, 2$$

See [ ] *H. B., J. N., in JMFM, 10 (2008), p.531-553.*

” inhomog. Navier–type boundary conditions ”

$$\mathbf{curl}\ \mathbf{u}^\nu \times \mathbf{n} \ = \ \mathbf{a}^\nu \text{ with } \mathbf{a}^\nu := \ \mathbf{curl}\ (\mathbf{u}^0 + \nu\mathbf{v}^0) \times \mathbf{n}$$

See [ ] *H. B., J. N., P. P., in DCDS - A (2010), 27, No. 4, p. 1353–1373.*

” inhomog. Navier’s boundary condition ”

$$[\mathbb{T}_d(\mathbf{u}^\nu)\cdot\mathbf{n}]_\tau + \gamma\mathbf{u}^\nu \ = \ \mathbf{c}^{\nu,\gamma} \text{ with } \mathbf{c}^{\nu,\gamma} := [\mathbb{T}_d(\mathbf{u}^0 + \nu\mathbf{v}^0)\cdot\mathbf{n}]_\tau + \gamma(\mathbf{u}^0 + \nu\mathbf{v}^0)$$

See [ ] *J. N., P. P., in a recent CRAS and in a forth’coming paper (Preprint 2010)*

Always the same structure $\mathbf{u}^\nu = \mathbf{u}^0 + \nu\, \mathbf{v}^0 + \nu\, \mathbf{w}^\nu$, $p^\nu = p^0 + \nu\, p_v^0 + \nu\, p_w^\nu$
as natural as possible, for $\mathbf{u}^\nu$, p^ν solving (5)(6)(7)(8) (*).

The key-points are : the first corrector $\mathbf{v}^0$ and
one of the three conditions (*), for example the third one.

$\mathbf{v}^0$ is chosen as the solution of the linear model (linear but hyperbolic, Euler-type)

$$\partial_t \mathbf{v}^0 + (\mathbf{u}^0 \cdot \nabla)\mathbf{v}^0 + (\mathbf{v}^0 \cdot \nabla)\mathbf{u}^0 = -\nabla p_v^0 + \Delta \mathbf{u}^0 \quad \text{in } Q_{T_0}, \tag{9}$$

$$\operatorname{div} \mathbf{v}^0 = 0 \quad \text{in } Q_{T_0}, \tag{10}$$

$$\mathbf{v}^0(.\,,0) = \mathbf{0} \quad \text{in } \Omega, \tag{11}$$

$$\mathbf{v}^0 \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{T_0}. \tag{12}$$

We recall the chosen boundary conditions for function $\mathbf{u}^\nu := \mathbf{u}^{\nu,\gamma}$ corresponding to

$$\mathbf{u}^{\nu,\gamma} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{T_0}, \tag{8}$$

$$[\mathbb{T}_d(\mathbf{u}^{\nu,\gamma}) \cdot \mathbf{n}]_\tau + \gamma \mathbf{u}^{\nu,\gamma} = c^{\nu,\gamma} \quad \text{on } \Gamma_{T_0}, \tag{*}$$

with $\mathbf{c}^{\nu,\gamma}$ given by $\mathbf{c}^{\nu,\gamma} := [\mathbb{T}_d(\mathbf{u}^0 + \nu\mathbf{v}^0) \cdot \mathbf{n}]_\tau + \gamma(\mathbf{u}^0 + \nu\mathbf{v}^0)$.

$\rightarrow$ So, $\mathbf{w}^{\nu,\gamma}$ and $p_w^{\nu,\gamma}$ must solve the following nonlinear system

$$\begin{aligned}
\partial_t \mathbf{w}^{\nu,\gamma} + \mathbf{u}^0 \cdot \nabla \mathbf{w}^{\nu,\gamma} + \mathbf{w}^{\nu,\gamma} \cdot \nabla \mathbf{u}^0 & \\
+ \nu \left(\mathbf{v}^0 \cdot \nabla \mathbf{w}^{\nu,\gamma} + \mathbf{w}^{\nu,\gamma} \cdot \nabla \mathbf{v}^0\right) & \\
+ \nu\, \mathbf{w}^{\nu,\gamma} \cdot \nabla \mathbf{w}^{\nu,\gamma} & \\
= - \nabla p_w^{\nu,\gamma} + \nu \Delta \mathbf{w}^{\nu,\gamma} + \nu(\Delta \mathbf{v}^0 - \mathbf{v}^0 \cdot \nabla \mathbf{v}^0) & \quad \text{in } Q_{T_0}, \qquad (13)
\end{aligned}$$

$$\operatorname{div} \mathbf{w}^{\nu,\gamma} = 0 \quad \text{in } Q_{T_0}, \qquad (14)$$

$$\mathbf{w}^{\nu,\gamma}(.\,,0) = \mathbf{0} \quad \text{in } \Omega. \qquad (15)$$

satisfying the boundary conditions on Γ_{T_0}

$$\mathbf{w}^{\nu,\gamma} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{T_0} \qquad (16)$$

$$\nu\big[\big(\nabla \mathbf{w}^{\nu,\gamma}\big)_s \cdot \mathbf{n}\big]_\tau + \nu\big[\big(\nabla \mathbf{w}^{\nu,\gamma\, T}\big)_s \cdot \mathbf{n}\big]_\tau + \gamma\, \mathbf{w}_\tau^{\nu,\gamma} = \mathbf{0} \quad \text{on } \Gamma_{T_0} \qquad (17)$$

(taking $\gamma = c_1 \nu$!)

Our intermediate results:

$\Rightarrow$ $(\mathbf{I})$ *Let* $\mathbf{u}^0$ *be the solution to the Euler system in* $L^\infty\big(0,T_0;\,\mathbf{W}^{4,2}_\sigma(\Omega)\big)$*, there exists a unique solution* $\mathbf{v}^0$ *in* $L^\infty\big(0,T_0;\,\mathbf{W}^{2,2}_\sigma(\Omega)\big)$ *to the Euler-type problem (9)–(12).*

p^0_v *is the associated pressure.*

And there exist a positive constant c_0*, depending on* Ω*,* T_0 *and on the norm* $|||\mathbf{u}^0|||_{1,4,2}$*, of course independent of* ν *and* γ*, to control the norm of* $\mathbf{v}^0$*.*

$\Rightarrow$ $(\mathbf{II})$ *Let* $c_1 > 0$ *be given, there exists* $\nu^* > 0$ *such that the nonlinear problem (13)–(17) has a unique solution* $\mathbf{w}^{\nu,\gamma} \in L^\infty\big(0,T_0;\,\mathbf{W}^{1,2}_\sigma(\Omega)\big) \cap L^2\big(0,T_0;\,\mathbf{W}^{2,2}(\Omega)\big)$ *for each* $\nu \in (0,\nu^*)$ *and* $\gamma \in (0, c_1\nu)$*.*

$p^{\nu,\gamma}_w$ *is the associated pressure.*

Solution $\mathbf{w}^{\nu,\gamma}$ *depends continuously on* ν *and* γ *in the norm* $|||\,.\,|||_{\infty;\,1,2} + |||\,.\,|||_{2;\,2,2}$*.*

And there exist positive constants c_2*,* c_3 *and* c_4*, depending on* Ω*,* T_0 *and on the norms* $|||\mathbf{u}^0|||_{1;\,4,2}$ *and* $|||\mathbf{v}^0|||_{\infty;\,2,2}$*, however all independent of* ν *and* γ*,*
such that

$$|||\mathbf{w}^{\nu,\gamma}|||_{\infty;\,0,2} \le c_2\,\nu, \quad |||\mathbf{w}^{\nu,\gamma}|||_{\infty;\,1,2} \le c_3\sqrt{\nu}, \quad |||\mathbf{w}^{\nu,\gamma}|||_{2;\,2,2} \le c_4. \qquad (18)$$

Theorem [on a family of solutions of the Euler or Navier–Stokes problem] *Suppose* $\mathbf{u}^* \in \mathbf{W}^{4,2}(\Omega) \cap \mathbf{L}^2_\sigma(\Omega)$ *and* $\mathbf{f} \in L^1\big(0,T;\,\mathbf{W}^{4,2}(\Omega)\big)$. *There exists* $T_0 \in (0,T]$ *such that to each* $c_1 > 0$ *there exists* $\nu^* > 0$

and a unique family $\{\mathbf{u}^{\nu,\gamma}\}$ *(for* $\nu \in [0,\nu^*)$ *and* $\gamma \in [0, c_1\nu]$*)*
of solutions to the Euler problem (1)–(4), if $\nu = 0$*,*
or to the Navier–Stokes problem (5)–(8)(*), if $\nu \in (0,\nu^*)$*,*
in $L^\infty\big(0,T_0;\,\mathbf{W}^{1,2}_\sigma(\Omega)\big) \cap L^2\big(0,T_0,\mathbf{W}^{2,2}(\Omega)\big)$.

Solution $\mathbf{u}^{\nu,\gamma}$ *depends continuously on* ν *and* γ *in the norm* $|||\,.\,|||_{\infty;\,1,2} + |||\,.\,|||_{2;\,2,2}$.

Solution $\mathbf{u}^{\nu,\gamma}$ *has the local in time structure* $\mathbf{u}^{\nu,\gamma} = \mathbf{u}^0 + \nu\mathbf{v}^0 + \nu\mathbf{w}^{\nu,\gamma}$*, where* $\mathbf{v}^0$ *is the solution of the linear problem (9)–(12) and* $\mathbf{w}^{\nu,\gamma}$ *is a solution of the nonlinear problem (13)–(16).*

The solutions $\mathbf{v}^0$ *and* $\mathbf{w}^{\nu,\gamma}$ *have the previously named properties; so the estimates (17) provide the rate of convergence of* $\mathbf{u}^{\nu,\gamma}$ *to* $\mathbf{u}^0$ *as* $\nu \to 0$.

Same structure for the associated pressure $p^{\nu,\gamma} = p^0 + \nu p^0_v + \nu p^{\nu,\gamma}_w$*, where* p^0_v *and* $p^{\nu,\gamma}_w$ *are the respective associated pressures to* $\mathbf{v}^0$ *and* $\mathbf{w}^{\nu,\gamma}$.

Thanks for your attention

Some open questions : $\longrightarrow$ Extended result for $\nu > \nu^*$?

$\longrightarrow$ Friction coefficient γ with $\gamma \leq c_1 \nu^\alpha$, $0 < \alpha < 1$?

$\longrightarrow$ and the challenging problem with Dirichlet's b.c. ?

It is well known that the choice of the no-slip complementary boundary condition is a source of various difficulties : T. Kato (1984) showed that the convergence of solutions of the Navier-Stokes equations to the solution of the Euler equations as $\nu \longrightarrow 0$ *is equivalent to the convergence to zero of the boundary layer strength.*

Concerning the presented result, see our papers,
J. Neustupa and P. Penel, a Note to CRAS Paris Ser. I 348 (2010), p.1093-1097
and/or the forth'coming detailled article "On the local in time existence of strong solutions of the Navier-Stokes equations with Navier's boundary conditions and the zero viscosity limit", Preprint 2010.